Linear Algebraic Groups, with Special Emphasis on the Classical Groups

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Abstract

This paper is an investigation of the nature and structure of linear algebraic groups over an algebraically closed field, and is particularly concerned with a detailed examination of the canonical examples of semisimple linear algebraic groups, namely the Classical Groups.

In the first instance, the general structure theory of reductive groups is exhaustively developed. All of this is by way of preparation, however, for the final chapter which uses the theory to set out in detail the structure of the Classical Groups. In particular, this final chapter computes in each case representative maximal tori and Borel subgroups, calculates the root system, and finally gives the Weyl group and Dynkin diagram. It is thus proven that the Classical Groups exhaust the infinite classes of irreducible root system, namely the classes A, B, C and D.

Candidate's Declaration

I declare that this thesis is my own work.

Michael Carmody

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Introduction

This paper is an investigation of the nature and structure of linear algebraic groups over an algebraically closed field, and is particularly concerned with a detailed examination of the canonical examples of semisimple linear algebraic groups, namely the Classical Groups. An algebraic group is an algebraic variety equipped with the additional structure and symmetries of a group, and as such it is an object which behaves very much both as a geometric and as an algebraic object. Indeed, the Classical Groups which in essence form the focus of our study are of course defined as invariants of certain linear transformations, but the approach we to use to investigate them is purely algebraic.

The treatment given here is only concerned with varieties over an algebraically closed field, K. This field is typically allowed to be of arbitrary characteristic, although because of the complications that fields of characterstic 2 can throw up, on occasion we will impose the assumption that $\mathrm{char} K \neq 2$.

The paper begins with the notion of an algebraic group, and thus presupposes that of an algebraic variety. An algebraic variety is a reduced scheme of finite type over the field K, or, equivalently, is the zero set of a finite collection of polynomials over K, equipped with the Zariski topology. We do not give a full treatment here of some of the results about varieties which are required in this paper, and in particular, we cite without proof some important results involving dimension and morphisms. The other main presupposition of this paper is the existence of a Lie algebra, associated to a linear algebraic group. The theory of Lie algebras is well known, even to undergraduates, and its results are easily accessible. As such, most of the standard results of Lie algebra theory are cited without proof.

The results here are typically taken from one of three main references, all with the title *Linear Algebraic Groups*, namely Borel's ([2]), Humphreys'([6]) and Springer's ([9]). For the most part, the proofs of all three books are similar, with minor deviations in approach here and there, not to mention minor deviations in clarity and accessibility. As such, the proofs in this paper are typically taken from one of these three references, albeit often expanded to ensure clarity and confidence. This approach also explains the great length of this paper – each of the results here is painstakingly proven, with attention to detail so as to ensure that the proofs are technically correct and follow logically from their predecessors.

Ultimately the aim of this paper is to derive the structure theory of reductive groups, and then to illustrate this theory with the practical examples of the Classical Groups. In Chapter 5, then, these examples are worked out in great detail. In particular, for each of the groups, we confirm their reductiveness and – except in the case of $GL(n, K)$ – their semisimplicity, then we calculate a representative Maximal Torus and Borel subgroup, go on to give their root space decomposition, and finally compute the Weyl groups and the Dynkin diagram in each semisimple case. The point is to emphasise the utility of the theory that has been so painstakingly developed in the preceeding chapters, and to give some concreteness to this theory, which is often dense and seemingly arcane. It is clear that the best way to understand the theory of reductive groups is to have these examples of the Classical Groups always at the front of one's mind.

Notation

Throughout this paper, the following notation is used.

A^c	The complement of the set U
$\mathrm{int}_X(A)$	The interior of the set in A
$Cl_X(A)$	The closure of the set in A
${}^t x$	The transpose of the matrix x
$\mathfrak{S}_n$	The symmetric group on n objects
$K[X]$	The affine algebra of the variety X
$\mathscr{I}(X)$	The ideal of functions vanishing on X
$\mathscr{V}(I)$	The subset of points which kill elements of the ideal I
$\mathscr{L}(X)$	The Lie algebra of the variety X
$\mathfrak{g}$	The Lie algebra of the algebraic group G

We also abbreviate some commonly used varieties, groups and algebras as follows:

$\mathbf{A}^n$	The n-dimensional affine variety
$\mathbb{P}^n$	The n-dimensional projective space
$\mathbb{P}(V)$	The projective space of lines in a vector space V
$\mathrm{Mat}(n, K)$	The algebra of $n \times n$ matrices
$D(n, K)$	The group of $n \times n$ invertible diagonal matrices
$\mathfrak{d}(n, K)$	The algebra of $n \times n$ diagonal matrices
$T(n, K)$	The group of $n \times n$ invertible upper triangular matrices
$\mathfrak{t}(n, K)$	The algebra of $n \times n$ upper triangular matrices
$U(n, K)$	The group of $n \times n$ invertible, upper triangular unipotent matrices

Chapter 1

Algebraic Groups

In this chapter we introduce our objects of study, namely the algebraic groups. An algebraic group is a group which is also a variety, and as such is a topological and geometric object, as well as being algebraic. Although the groups we examine have associated topologies, they are typically not topological groups in the technical sense, since the underlying topology is the Zariski topology.

Ultimately, this paper is only concerned with linear algebraic groups, rather than the other main branch of algebraic groups, namely the abelian algebraic groups. The linear algebraic groups are those which have an affine underlying variety, that is, a variety which is isomorphic to a Zariski closed subset of affine space.

This chapter also introduces the canonical examples of linear algebraic groups which ought to be kept in front of mind, namely the Classical Groups. A few brief observations about the Classical are also made, for example, their connectedness, but for the most part discussion about these groups is held off until Chapter 5, where they are analysed thoroughly.

1.1 Beginnings

We start by defining the objects, and then we introduce for the first time the most important examples, namely the so called 'Classical Groups'. These groups serve as the canonical examples, and it is ultimately a study of their structure which we are concerned with. That study will have to wait until Chapter 5, however, after the groundwork has been sufficiently prepared.

We take the notions of *affine variety* and the generalisation *variety* from [6]. That is, for K an algebraically closed field, an *affine variety* is a subset of K^n consisting of common zeros of a finite collection of polynomials $f_1, \dots, f_k \in R$, while a variety is a ringed space $(X, \mathcal{O}_X)$ where X is a noetherian topological space whose diagonal $\Delta(X)$ is closed in $X \times X$ under the Zariski topology, and where any point $x \in X$ has an open neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine variety over K. We often simply write X for $(X, \mathcal{O}_X)$

when the sheaf of rings is clear. Note also that an affine variety is clearly a variety with the sheaf of rings of polynomial functions on open sets.

1.1.1 Definition

Definition 1.1.1.1. Let G be a variety over a field K. Suppose further that G is a group such that the following two maps which define the group structure

$$\begin{aligned} \mu : & G \times G \longrightarrow G, \quad \mu(x,y) = xy \\ \imath : & G \longrightarrow G, \quad \imath(x) = x^{-1} \end{aligned}$$

are morphisms of varieties. Then G is called an *algebraic group.* If G is an affine variety, than we call G a *linear algebraic group* or an *affine algebraic group.*

Unless otherwise indicated, we will assume that all algebraic groups under discussion are affine. We use the notation of [6]: given an affine variety X over a field K, $K[T]$ denotes the polynomial ring in one indeterminate, and $K[X]$ denotes the algebra of polynomial functions on X, that is, the K-algebra $K[X] = K[T]/\mathscr{I}(X)$, where

$$\mathscr{I}(X) = \{f \in K[T] \mid \forall x \in X, f(x) = 0\}$$

1.1.2 Some Examples

K^n equipped with the sheaf of polynomial functions on open sets (in the Zariski topology) is clearly an affine variety, and is denoted $\mathbf{A}^n$.

Example 1. Consider the affine variety $\mathbf{A}^1 = K$. This is a group under addition. Since a morphism of affine varieties $X \to \mathbf{A}^1$ is precisely a polynomial function on X, then certainly

$$\begin{aligned} \mu : \mathbf{A}^1 \times \mathbf{A}^1 &\longrightarrow \mathbf{A}^1 \\ (x,y) &\longmapsto x+y \end{aligned}$$

is a morphism of affine varieties, and similarly for $\imath : \mathbf{A}^1 \to \mathbf{A}^1$. Thus $\mathbf{A}^1$ is an algebraic group.

We have a homomorphism

$$\mu^* : K[T] \longrightarrow K[T] \otimes_K K[T] \cong K[T,U]$$

Indeed, with $f = T \in K[T]$, we have

$$\mu^*(f)(T,U) = f \circ \mu(T,U) = f(T+U) = T+U.$$

Due to the importance of this algebraic group, we give it the special notation $\mathbb{G}_a$. Certainly $\dim \mathbb{G}_a = 1$.

Example 2. Consider the set $\mathbf{A}^1 - \{0\} = K^*$. Note that this set is a principal open set of $\mathbf{A}^1$. Furthermore, K^* is a group under multiplication. Since $\mu : K^* \times K^* \to K^*$ is a polynomial function on $K^* \times K^*$, the multiplication map is a morphism of algebraic varieties, and similarly for the inverse map.

Due to the importance of this algebraic group, we give it the special notation $\mathbb{G}_m$. We should also note here that

$$K[\mathbb{G}_m] = (K[T])[1/T] = K[T, T^{-1}]$$

and so $\dim \mathbb{G}_m = 1$.

It is a fact, which we will not prove here, that $\mathbb{G}_a$ and $\mathbb{G}_m$ are the only irreducible algebraic groups of dimension 1. In fact, the only difficult case is that of a unipotent algebraic group (a concept which we will define explicitly in Definition 2.2.4.1), over a field of characteristic $p \neq 0$. The other cases turn out to be elementary enough applications of some of the results in our Chapter 2. However, the case of unipotent subgroups over nonzero characteristic fields requires more ingenuity, and certainly more algebra. The proof is quite involved, but can be found in elegant and self-contained detail in §2.6 of [9]. An alternate approach can be found as Theorem 10.9 in [2]. This proof hinges on a fact which we will prove later, as Theorem 3.3.2.2, which explicitly describes the automorphism group of $\mathbb{P}^1$. One then uses an embedding of the one dimensional group in $\mathbb{P}^1$, which is essentially a geometric fact, to prove the result.

Example 3. Let

$$\begin{aligned} \mathrm{Mat}(n, K) &= \{n \times n \text{ matrices over } K\} \\ GL(n, K) &= \text{ general linear group } = \{x \in \mathrm{Mat}(n, K) \mid \det(x) \neq 0\} \end{aligned}$$

Now, certainly

$$K[\mathrm{Mat}(n, K)] = K[T_{ij}]_{1 \leq i,j \leq n}$$

and given that $GL(n, K)$ is a principal open set, we have

$$K[GL(n, K)] = \big(K[T_{ij}]\big)[1/\det T_{ij}] = K[T_{ij}, 1/\det].$$

Furthermore

$$\begin{aligned} \mu^*(T_{ij}) &= \sum_{h=1}^{n} T_{ih} \otimes T_{hj} \\ \imath^*(T_{ij}) &= (i, j)^{th} \text{ entry of } (T_{ij})^{-1} \\ e(T_{ij}) &= \delta_{ij} \qquad \text{(obvious)} \end{aligned}$$

We should also note that $\mathbb{G}_m = GL(1, K)$.

Proposition 1.1.2.1. *Given two algebraic groups G, G', the product variety $G \times G'$ forms an algebraic group by giving it the usual group product structure.*

Proof. Firstly, we need that the multiplication map on $(G \times G') \times (G \times G')$ is a morphism of varieties. But this is equal to $\mu \times \mu'$ where μ, μ' are the multiplication maps for G, G' respectively. It is a fact, which arises from Proposition 2.4 of [6], that $\mu \times \mu'$ is a morphism of varieties. Similarly, the inversion map on $G \times G'$ is equal to $\imath \times \imath'$, and so is also a morphism of varieties. □

Note that the product of algebraic groups as in Proposition 1.1.2.1 are not topological groups, given that they have the Zariski rather than product topology.

Example 4. The group of $n \times n$ diagonal matrices $D(n, K)$ is isomorphic to the product of n copies of $\mathbb{G}_m$. In symbols,

$$D(n, K) \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$$

Similarly,

$$\mathbf{A}^n \cong \mathbb{G}_a \times \cdots \times \mathbb{G}_a$$

Definition 1.1.2.2. A morphism of varieties $\varphi : G \to G'$ is called a *homomorphism of algebraic groups* if φ is also a group homomorphism. φ is an *isomorphism of algebraic groups* if it is an isomorphism of groups and of varieties. Similarly for *automorphisms.*

Proposition 1.1.2.3. *Let G be an algebraic group, with $g \in G$. Then the maps*

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto gx \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto xg \end{aligned}$$

are isomorphisms of varieties.

Proof. Since for given $g \in G$, we have $xg, gx \in K[G]$, the maps are morphisms of varieties. Moreover, by replacing g above with g^{-1}, it follows that the maps are isomorphisms, as required. □

Definition 1.1.2.4. When the target group under a morphism of algebraic groups φ is $GL(n, K)$, we say that φ is a *rational representation.*

Proposition 1.1.2.5. *Any closed subgroup G of $GL(n, K)$ (in the Zariski topology) is a linear algebraic group.*

Proof. This is a special case of the more general fact that a closed subgroup of a linear algebraic group is again a linear algebraic group. Indeed, a closed subset of an affine variety is certainly itself an affine variety. Moreover, the morphisms which define the group operations of an algebraic group induce morphisms of varieties when restricted to a closed subgroup. This is because morphisms of affine varieties restricted to closed subsets are themselves morphisms of affine varieties. □

Example 5. Any finite subgroup of $GL(n, K)$ is a linear algebraic group, since points are closed in the Zariski topology.

Remark. In this thesis, we focus on linear algebraic groups, terminology which we will see later derives from their deep connection to the general linear group. There is another class of algebraic groups, however, the so called 'non-linear' algebraic groups. These have non-affine underlying varieties. The canonical examples of non-linear algebraic groups are the *elliptic curves*, namely closed subsets of $\mathbb{P}^2$, which are certainly not affine. For example, if $\mathrm{char} K \neq 2, 3$, then points on the curve $y^2 = x^3 - ax - b$ form an abelian group.

From this point on, we will usually assume that an algebraic group is affine.

1.2 Classical Algebraic Groups

We have already met the most important linear algebraic group, namely $GL(n, K)$. Here we will introduce certain subgroups of $GL(n, K)$ which constitute the other key examples of algebraic groups. This is just a tease, however, and we have to wait until Chapter 5 for a more detailed analysis of their structure.

The parameter l in the following describes the dimension of the (closed) subgroup of the diagonal matrices in the groups under discussion. Later we will come to learm more about the importance of this subgroup.

The *special linear group* $SL(l + 1, K)$

$SL(l+1, K)$ is defined to be the group which consists of matrices of determinant equal to 1 in $GL(l + 1, K)$. This is certainly a group, and it is closed, since

$$SL(l + 1, K) = \{x \in GL(l + 1, K) \mid \det(x) - 1 = 0\}$$

The *symplectic group* $Sp(2l, K)$

$Sp(2l, K)$ is defined to be the group which consists of all matrices $x \in GL(2l, K)$ satisfying

$$^{t}xsx = s \tag{1.1}$$

where

$$s = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 0 & & 0 & 1 \\ 0 & & 1 & 0 \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & 0 & 0 \end{pmatrix}$$

This is a group, since it is easily shown to be closed under multiplication and taking of inverses. Moreover, since equation (1.1) imposes polynomial conditions on x, $Sp(2l, K)$ is closed in $GL(2l, K)$. Finally observe that $\det s = 1$ for all l, and so elements of $Sp(2l, K)$ necessarily have determinant ± 1.

The *special orthogonal group* $SO(2l+1, K)$

We impose the assumption firstly that char$K \neq 2$. $SO(2l+1, K)$ is defined to be which consists of all $x \in SL(2l+1, K)$ which satisfy

$$^{t}xsx = s$$

where

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix}$$

and J is as above. Again, this condition defines a closed subgroup of $GL(2l+1, K)$.

The *special orthogonal group* $SO(2l, K)$

Again we assume char$K \neq 2$. $SO(2l, K)$ is defined to consist of all $x \in SL(2l+1, K)$ which satisfy

$$^{t}xsx = s$$

where

$$s = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$

and J is as above. Again, this condition defines a closed subgroup of $GL(2l, K)$.

We will return to the Classical Algebraic Groups in much greater detail in Chapter 5.

1.3 Preliminary Results

We begin our study of the theory proper with some immediate results, especially concerning the topological structure of algebraic groups.

1.3.1 Topology

Proposition 1.3.1.1. *Let G be an algebraic group. Then*

1. *There is only one irreducible component which contains the identity $e \in G$. Denote this component by G°.*

2. *G° is a closed, normal subgroup of finite index in G.*

3. *The cosets of G° in G are the irreducible components of G. In particular, G° is the irreducible component for e.*

4. *The irreducible components of G are also the connected components of G.*

5. *Any closed subgroup $H < G$ of finite index in G has the property $G^\circ \subset H$.*

Proof. (1): Let $X_1, \ldots, X_m \subset G$ be the distinct irreducible components of G which contain e. Now, by Proposition 1.1.2.3, the product morphism

$$\begin{aligned}\varphi : X_1 \times \cdots \times X_m &\longrightarrow G\\ (x_1, \ldots, x_m) &\longmapsto x_1 \cdots x_m\end{aligned}$$

is an isomorphism of varieties, and so is continuous. Furthermore, $X_1 \times \cdots \times X_m$ is irreducible since each X_i is, and by continuity, $\mathrm{im}\varphi$ is irreducible, too. But $\varphi(e, e, \ldots, e) = e \in \mathrm{im}\varphi$ which means $\mathrm{im}\varphi = X_j$ for some j. But, for each element $x \in X_i$, we have $\varphi(e, \ldots, x, \ldots e) = x_i \in \varphi$ so we deduce that $X_1 = \cdots = X_m = G^\circ$.

(2): Since G° is irreducible, so is $\overline{G^\circ}$. But $e \in \overline{G^\circ}$, and so by (1), $\overline{G^\circ} = G^\circ$. From the first part of the proof, we know that G° is closed under multiplication, and by a similar argument, because $\imath$ is an isomporphism of algebraic varieties, we have $\imath(G^\circ) = G^\circ$, and so it is closed under taking of inverses, and is therefore a subgroup. Similarly, for any $x \in G$ we have $e \in xG^\circ x^{-1}$ so $G^\circ = xG^\circ x^{-1}$, which is to say that G° is normal. Now, for each $x \in G$, we have that the coset xG° is closed and irreducible by Proposition 1.1.2.3. Moreover, these cosets are pairwise distinct and their union is G. Since there are only finitely many irreducible components of G, there can be only finitely many of these cosets, and this implies that G° is of finite index in G.

(3): For each $x \in G$, we have that $x : G^\circ \to xG^\circ$ is an isomorphism of varieties, and so xG° is closed and irreducible. But G is the union of these finitely many, pairwise distinct, irreducible subsets. The irreducible components of a neotherian space like G are unique, and so it follows that it is precisely these cosets which are the irreducible components of G.

(4): From (3), we know that the irreducible components of G are the cosets of G°. It is an easy enough fact that if the irreducible components of a space are pairwise disjoint, then they are the connected components of that space, as required here.

(5): Given a closed subgroup H of finite index in G, the same argument used in (3) gives us that each of the cosets xH are closed in G. But G itself is a finite union of these cosets, and so each xH is the complement of a the union of the other cosets, which is closed. Therefore, xH is open in G, which means xH is a clopen set. Since $e \in H$, we have $H \cap G^\circ \neq \emptyset$, and by the fact, proved below, that any clopen subset is a union of connected components, we have $G^\circ \subset H$, as required. □

In the proof of Proposition 1.3.1.1 (5), I used the following fact:

Lemma 1.3.1.2. *Any clopen set $U \subset Z$ is a union of connected components of Z.*

Proof. Certainly we can write Z as the disjoint union of its connected components, say $Z = \bigcup Z_\alpha$. Then

$$U = U \cap Z = U \cap \bigcup Z_\alpha = \bigcup U \cap Z_\alpha \tag{1.2}$$

But, for each α, we have

$$Z_\alpha = (Z_\alpha \cap U) \cup (Z_\alpha \cap U^c) \tag{1.3}$$

where U^c is the complement of U in Z. But since U is clopen, we have that U is open in Z, and so is U^c. Therefore, in equation (1.3), we have written a connected space as the union of two open subsets. Therefore, either $Z_\alpha \cap U = \emptyset$ which means $Z_\alpha \subset U$ or $Z_\alpha \cap U^c$ which maens $Z_\alpha \subset U^c$. It follows that each of the terms in the left hand side of equation (1.2) is either empty, or equal to Z_α, that is, we have written U as a union of connected components, as required. $\square$

Proposition 1.3.1.1 tells us, then, that for algebraic groups, the concepts of irreduciblility and connectedness are equivalent. Note further that (2) tells us that G/G° is a finite group.

Definition 1.3.1.3. We call G° the *identity component* of G. Moreover, we call an algebraic group G *connected* (rather than irreducible) when $G = G^\circ$.

Example 6. The groups $\mathbb{G}_m$ and $\mathbb{G}_a$ and $GL(n, K)$ are all connected.

Example 7. The group $D(n, K)$ is connected. So is the group of upper triangular $n \times n$ matrices, which we will denote by $T(n, K)$, and likewise the group of upper triangular $n \times n$ matrices of determinant 1, which we will denote by $U(n, K)$.

Proposition 1.3.1.4. $\dim G = \dim G^\circ$.

Proof. This is simply by definition of dimension, combined with Proposition 1.3.1.1. $\square$

A little exercise about connectedness gives the following:

Proposition 1.3.1.5. *Let G be a connected algebraic group, and N a finite normal subgroup. Then $N \subset Z(G)$, where $Z(G)$ denotes the centre of G.*

Proof. Let $n \in N$, and define a morphism

$$\begin{aligned} \alpha : G &\longrightarrow N \\ g &\longmapsto gng^{-1} \end{aligned}$$

Since α is clearly a morphism of algebraic groups, we know that $\mathrm{im}(\alpha)$ is connected. But given that N is finite, the only connected components are points, which is to say $\mathrm{im}\alpha = \{n'\}$ for some $n' \in N$. But $\alpha(e) = n$, so $n' = n$, which is to say, for all $n \in N$, we have $n = gng^{-1}$ for all $g \in G$, as required. $\square$

Proposition 1.3.1.6. *Let G be a connected algebraic group. Then any nonempty open subset $U \subset G$ is dense in G.*

Proof. This is simply by definition of irreducibility, since if $V \subset G$ open, $U \cap V \neq \emptyset$, and therefore U is dense in G. Therefore, the result follows from Proposition 1.3.1.1, which established equivalence between connectedness and irreducibility in the case of algebraic groups. $\square$

Lemma 1.3.1.7. *Let U, V be dense open subsets of an algebraic group G. Then $G = U.V$*

Proof. Let $x \in G$, and consider the subset

$$xV^{-1} = \{xy^{-1} \mid y \in V\}$$

It is clear that xV^{-1} is open and dense in G, and therefore $xV^{-1} \cap U \neq \emptyset$. Let $z \in xV^{-1} \cap U$. That is, there is a $y \in V$ such that $xy^{-1} = z$, whence $x = zy$, where $z \in U, y \in V$, as required. $\square$

Remark. In the above lemma, if G is connected we can use Proposition 1.3.1.6 and then we require only that U, V are open and nonempty.

Recall that a set is said to be *locally closed* if it is the intersection of an open set and a closed set, and *constructible* if it is a finite union of locally closed sets.

Lemma 1.3.1.8. *If $Y \subset X$ is constructible, then there exists a set $U \subset Y$ such that U is open and dense in $\bar{Y}$.*

Proof. Proof. Since Y is constructible, there exist finitely many sets U_i open in X and sets F_i closed in X, where $1 \leq i \leq n$, such that

$$Y = \bigcup_{1 \leq i \leq n} U_i \cap F_i$$

We define an open set

$$U = \bigcup_{1 \leq i \leq n} U_i \cap \mathrm{int}_{\bar{Y}}(F_i)$$

where $\mathrm{int}_{\bar{Y}}(F_i)$ here denotes the interior in $\bar{Y}$ of F_i. Now certainly U is contained in Y and is open in $\bar{Y}$. Furthermore,

$$\mathrm{Cl}_{\bar{Y}}(U) = \bigcup_{1 \leq i \leq n} \mathrm{Cl}_{\bar{Y}}(U_i \cap \mathrm{int}_{\bar{Y}}(F_i)) = \bigcup_{1 \leq i \leq n} \mathrm{Cl}_{\bar{Y}}(U_i) \cap F_i = \overline{Y}$$

and so U is dense in $\overline{Y}$. $\square$

$\square$

Proposition 1.3.1.9. *Let $H < G$ be a subgroup. Then*

1. *$\overline{H}$ is a subgroup of G*
2. *If H contains a nonempty open subset of $\overline{H}$, then $H = \overline{H}$.*
3. *If H is constructible, then $H = \overline{H}$.*

Proof. For (1), let $x \in H$. Certainly $H = xH \subset x\overline{H}$. Since translation is an isomorphism of varieties by Proposition 1.1.2.3, we know that $x\overline{H}$ is closed, and therefore $\overline{H} \subset x\overline{H}$, which in turn tells us $x^{-1}\overline{H} \subset \overline{H}$, or, equally, $H.\overline{H} \subset \overline{H}$. Therefore, for any $x \in \overline{H}$, we have $Hx \subset \overline{H}$, and so, taking closures, we have

$\overline{Hx} = \overline{H}x \subset \overline{H}$, whence $\overline{H}.\overline{H} \subset \overline{H}$, and so $\overline{H}$ is closed under multiplication. Similarly, given that inversion is a homeomorphism, we have $\overline{H}^{-1} = \overline{H}$ and $\overline{H} = \overline{H^{-1}}$, which shows that $\overline{H}$ is closed under inversion, and so is a subgroup, so (1) is proved.

For (2), let $U \subset H$ be open in $\overline{H}$. Since translation by an element of the group H is a homeomorphism, it follows that xU is open in $\overline{H}$ for all $x \in H$. Since H is a group, it is also clear that $\bigcup_{x \in H} xU = H$, and so H is open in $\overline{H}$, and of course is dense. Now applying Lemma 1.3.1.7 tells us $H.H = H = \overline{H}$.

Finally, (3) follows from (2). Firstly, observe that if a set is constructible, it is easy to see that it contains a subset dense in its closure. That is, H contains an open subset of $\overline{H}$ which is dense in $\overline{H}$, but by (2) this same set is dense in H, and so we must conclude that $H = \overline{H}$, as required. □

The following corollary is useful later:

Corollary 1.3.1.10. *Let A, B be closed subgroups of an algebraic group G. If B normalises A, then AB is a closed subgroup of G.*

Proof. Since $B \subset N_G(A)$, then AB is a subgroup. Indeed, if $a_1b_1, a_2b_2 \in AB$, then $a_1b_1a_2b_2 = a_1a'b'b_2$ for some $a' \in A, b' \in B$, and so AB is closed under multiplication. Similarly, $(a_1b_1)^{-1} = b_1^{-1}a_1^{-1} = a_1'b_1^{-1}$ for some $a' \in A$, so AB is closed under inversion, and is therefore a subgroup.

Now, AB is clearly the image of the map

$$\mu : A \times B \longrightarrow G$$

where μ is the product morphism. Since A, B are closed, $A \times B$ is also closed, and is therefore constructible, and in turn, so is the image AB under the above morphism. By Proposition 1.3.1.9 (3), we have $AB = \overline{AB}$, as required. □

1.3.2 Morphisms

Proposition 1.3.2.1. *Let $\varphi : G \to G'$ be a morphism of algebraic groups. Then*

1. $\ker \varphi$ *is a closed, normal subgroup of G.*
2. $\mathrm{im}\varphi$ *is a closed subgroup of G'.*
3. $\varphi(G^\circ) = [\varphi(G)]^\circ$
4. $\dim G = \dim \ker \varphi + \dim \mathrm{im}\varphi$

Proof. For (1), note that $\{e\}$ is closed and $\ker \varphi = \phi^{-1}\{e\}$, and so is closed; it is an elementary result of group theory that it is a normal subgroup. For (2), note firstly that morphisms send constructible sets to constructible sets, and in particular $\varphi(G)$ is constructible. By Proposition 1.3.1.9(3), then, $\varphi(G)$ is closed in G'. For (3), we have that $\varphi(G^\circ)$ is closed, by (2), and is irreducible since morphisms send irreducible sets to irreducible sets. This gives us that $\varphi(G^\circ) \subset \varphi(G)^\circ$. Now G° is of finite index in G, which means that $\varphi(G^\circ)$ is of finite index in $\varphi(G)$, and so Proposition 1.3.1.1 tells us that $\varphi(G)^\circ \subset \varphi(G^\circ)$, as required.

For (4), we need to refer to Theorem 4.3 of [6]. This theorem is about the dimension of irreducible varieties, but (3) above, we can assume that G and $\varphi(G)$ are connected. In particular, the theorem says that there exists an element $y \in \varphi(G)$ such that $\dim G - \dim \varphi(G) = \dim \varphi^{-1}(y)$. But, if we take $y \in G$ such that $\varphi(y) = x$, then the map $\ker\varphi \to x(\ker\varphi)$ is an isomorphism by Propositoin 1.1.2.3. But $\ker\varphi = \varphi^{-1}(e)$ and $x(\ker\varphi) = \varphi^{-1}(x)$, and so $\dim G - \dim \varphi(G) = \ker\varphi$, as required. $\square$

Example 8. The map

$$\begin{aligned} \det : GL(n,K) &\longrightarrow \mathbb{G}_m \\ x &\longmapsto \det(x) \end{aligned}$$

is polynomial in the entries of $x \in GL(n,K)$, and so is a surjective morphism of algebraic groups. Moreover, $\ker\det = SL(n,K)$, and so $\dim SL(n,K) = n^2 - 1$.

The following lemma will also prove useful later:

Lemma 1.3.2.2. *Let H, G be algebraic groups. If $H \subset G$, then $H^\circ \subset G^\circ$.*

Proof. Consider the inclusion morphism

$$\begin{aligned} \varphi : H &\hookrightarrow G \\ h &\longmapsto h \end{aligned}$$

We know from Proposition 1.3.2.1 that $H^\circ = \varphi(H^\circ) = \varphi(H)^\circ$. For convenience, we will denote this group by $H^\circ = G'$. We now claim G' lies entirely within one of the irreducible components of G. Let $G = G_1 \cup \cdots \cup G_m$ be the decomposition of G into irreducible components, and let $G'_i = G' \cap G_i$. Now, G_i is closed in G by definition, and since G' is closed in G by Proposition 1.3.2.1, it follows that each G'_i is closed in G. Moreover, $G' = \cup G'_i$, which implies G' is a finite union of disjoint closed sets. Since G' is connected, it follows that all but one G'_i must be empty, which implies $G' \subset G_i$ for some i. However, $e \in G'$, since $e \in H$, and so $e \in G_i$, which implies $G_i = G^\circ$. We have shown, therefore, that $H^\circ = G' \subset G^\circ$, as required. $\square$

Definition 1.3.2.3. Let $M \subset G$ be a subset of an algebraic group, G. Denote by $\mathcal{A}(M)$ the intersection of all closed subgroups of G which contain M, i.e.

$$\mathcal{A}(M) = \bigcap_{\substack{H=\overline{H}<G \\ M\subset H}} H$$

This is the smallest closed subgroup of G which contains M, and is called the *group closure* of M.

Proposition 1.3.2.4. *Let G be an algebraic group, I an index set, $\{X_i\}_{i\in I}$ a family of irreducible varieties, and $f_i : X_i \to G$ a family of morphisms such that*

$$e \in Y_i = f_i(X_i) \quad \forall i \in I.$$

Set $M = \bigcup_{i\in I} Y_i$. Then

1. *$\mathcal{A}(M)$ is a closed, connected subgroup of G*
2. *For some finite sequence $a = (a_1, \dots a_m), a_j \in I$, we have*

$$\mathcal{A}(M) = Y_{a_1}^{\epsilon_1} \cdots Y_{a_m}^{\epsilon_m}, \qquad \epsilon_j = \pm 1$$

In order to prove Proposition 1.3.2.4, we need the following lemma:

Lemma 1.3.2.5. *Given topological spaces X, Y, subsets $A \subset X, B \subset Y$ and a continuous function $f : X \to Y$, then*

$$f(A) \subset B \Longrightarrow f(\overline{A}) \subset \overline{B}$$

Proof. If $f(A) \subset B$ then certainly we have $\overline{f(A)} \subset \overline{B}$, so it suffices to show that $f(\overline{A}) \subset \overline{f(A)}$. We define the set $F = f^{-1}\left(\overline{f(A)}\right)$, which is closed in X since f is continuous. Suppose $x \in A$. Then $f(x) \subset f(A) \subset \overline{f(A)}$. Therefore $x \in f^{-1}\left(\overline{f(A)}\right) = F$, and so $A \subset F$. Since F is closed, this tells us that $\overline{A} \subset F$, and so $f(\overline{A}) \subset f(F)$. But it is easy to see that $f(F) = f\left(f^{-1}\left(\overline{f(A)}\right)\right) \subset \overline{f(A)}$, and so $f(\overline{A}) \subset \overline{f(A)}$, as required. □

Proof of Proposition 1.3.2.4. It is easy to see that we do not alter the group $\mathcal{A}(M)$ if we extend the index set I to include morphisms h_i which are defined

$$\begin{aligned} h_i : X_i &\longrightarrow G \\ x &\longmapsto f(x)^{-1} \end{aligned}$$

Now, for each finite sequence $a = (a_1, \dots, a_n)$ such that $a_j \in X_j$ for $j \in I$, we define a set

$$Y_a = Y_{a_1} \cdots Y_{a_n}$$

Note that Y_a is the image of

$$X_{a_1} \times \cdots \times X_{a_n} \xrightarrow{(f_1,\dots,f_n)} Y_{a_1} \times \cdots \times Y_{a_n} \xrightarrow{\gamma} Y_a$$

where γ is simply an accumulation of product morphisms. This tells us that Y_a is constructible and also that it is irreducible, and therefore that $\overline{Y}_a$ is also irreducible. Finally, we observe that $e \in \overline{Y}_a$. Note, then, that we have a collection of closed, irreducible subsets of a variety G, and therefore we have a maximal one, by the fact that G, like all varieties, is a noetherian topological space under its Zariski topology. Call this maximal element $\overline{Y}_a$, say, defined by the finite sequence a.

Now , given finite sequences b, c of elements of I, we can construct for any $y \in Y_c$ a continuous map

$$\begin{aligned} Y_b &\longrightarrow Y_{(b,c)} \\ x &\longmapsto xy \end{aligned}$$

where (b, c) denotes the sequence obtained by concatenating b and c. By Lemma 1.3.2.5, we see that $\overline{Y}_b$ maps into $\overline{Y}_{(b,c)}$, which tells us that $\overline{Y}_b Y_c \subset \overline{Y}_{(b,c)}$. This in turn means that for any $x \in \overline{Y}_b$, we can define a continuous map

$$\begin{aligned} Y_c &\longrightarrow \overline{Y}_{(b,c)} \\ y &\longmapsto xy \end{aligned}$$

and another application of Lemma 1.3.2.5 then tells us that $\overline{Y}_c$ maps into $\overline{Y}_{(b,c)}$, from which we can conclude that, for arbitrary finite sequences b and c, we have

$$\overline{Y}_b \overline{Y}_c \subset \overline{Y}_{(b,c)} \tag{1.4}$$

Now, since $\overline{Y}_a$ is maximal, for any sequence b we have

$$\overline{Y}_a \subset \overline{Y}_a \overline{Y}_b \subset \overline{Y}_{(a,b)} \subset \overline{Y}_a \tag{1.5}$$

where the first inclusion is by the fact that $e \in \overline{Y}_b$, and the second inclusion is by equation (1.4). If we set $b = a$, then we get $\overline{Y}_a \overline{Y}_a = \overline{Y}_a$, that is, $\overline{Y}_a$ is stable under multiplication.

Similarly to equation (1.5), we have, for arbitrary b,

$$\overline{Y}_b \subset \overline{Y}_a \overline{Y}_b \subset \overline{Y}_{(a,b)} \subset \overline{Y}_a \tag{1.6}$$

We said that we expanded I sufficiently, too, such that for some b we have $Y_a^{-1} = Y_b$, and so $\overline{Y}_a^{-1} \subset \overline{Y}_a$, that is, $\overline{Y}_a$ is stable under inversion. This then shows that $\overline{Y}_a$ is a subgroup of G. Note also that equation (1.6) tells us that $Y_b \subset \overline{Y}_a$ for all b, and in particular $Y_i \subset \overline{Y}_a$ for all $i \in I$. From this we conclude that $\overline{Y}_a = \mathscr{A}(M)$, so $\mathscr{A}(M)$ is closed, and since $\overline{Y}_a$ is irreducible, that is, connected, we have proved (1).

For (2), we first observe that since we noted that Y_a is constructible, it contains an open subset U which is dense in $\overline{Y}_a$. Now we use Lemma 1.3.1.7 to note that $U \cdot U = \overline{Y}_a$, and so

$$\overline{Y}_a = U \cdot U \subset Y_a \cdot Y_a = Y_{(a,a)} \subset \overline{Y}_{(a,a)} \subset \overline{Y}_a$$

and so $\overline{Y}_a = Y_{(a,a)}$, which means the sequence (a, a) satisfies (2). □

Corollary 1.3.2.6. *Let G be an algebraic group, $Y_i, i \in I$ a family of closed, connected subgroups of G which generate G (as an abstract group). Then G is connected.*

Proof. If the Y_i generate G as a group, then certainly $\mathscr{A}(M) = G$ where $M = \bigcup Y_i$. If we take f_i simply as identity morphisms and apply Proposition 1.3.2.4 with $X_i = Y_i$ the irreducible varieties, then we can conclude that G is connected. $\square$

Corollary 1.3.2.7. *Let G be an algebraic group with H, K closed subgroups, one of which is connected. Then (H, K) is connected.*

Proof. Suppose H is connected. Then we use Proposition 1.3.2.4, setting $I = K$, and all $X_i = H$. If we then define, for each $y \in K$,

$$f_y(x) = xyx^{-1}y^{-1}$$

then $e \in f_y(H) = Y_y$, and clearly $\mathcal{A}(M) = (H, K)$ in the notation of Proposition 1.3.2.4. This then gives us that (H, K) is connected, as required. $\square$

We now use these results to show that some important algebraic groups are connected.

Proposition 1.3.2.8. *Each of the Classical Groups given in §1.2 is connected.*

Proof. It is an easily established fact that $SL(n, K)$ is generated (as an abstract group) by the subgroups U_{ij}, $(i \neq j)$, which consist of the matrices with 1's on the diagonal, an arbitrary entry in the $(i, j)^{\text{th}}$ entry, and 0's everywhere else. The next step is to note that the map

$$\alpha_{ij} : \mathbb{G}_a \longrightarrow U_{ij}$$

which sends an element of $x \in \mathbb{G}_a$ to the matrix which has x as its $(i, j)^{\text{th}}$ entry, 1's on the diagonal, and 0's everywhere else is an isomorphism of algebraic groups. That it is a isomorphism of varieties is clear enough, and to show that it is a group homomorphism simply note that multiplication in U_{ij} is a matter of adding the $(i, j)^{\text{th}}$ entries. Now, since $\mathbb{G}_a$ is connected, this tells us that the U_{ij}'s are, too, and hence we apply Corollary 1.3.2.6 to conclude that $SL(n, K)$ is connected.

The proof to show that $Sp(2n, K)$ is connected is identical, once it is established that it is generated by closed, connected groups, which is essentially an elementary algebraic result. Indeed, we cite Lemma 1 in §6.9 of [7], which says that $Sp(2n, K)$ is generated by *symplectic transvections.* Each transvection is an endomorphism along a particular direction, and all the transvections in a particular direction form a subgroup which is isomorphic to $\mathbb{G}_a$, and so we can repeat the above argument.

Similarly, the proof to show that $SO(n, K)$ is connected is the same again, once we know that any orthogonal transformation is generated by symmetries; see, for example, Theorem 6.12 of [7]. Again, each symmetry has an associated direction, and the symmetries in a given direction form a subgroup which is isomorphic to $\mathbb{G}_a$. Thus, the same argument as given above show that $SO(n, K)$ is connected.

Remark. The proofs which show that the remaining Classical Groups, namely $SO(n, K)$ and $Sp(2n, K)$ are connected are very similar, and depend on connected generating subgroups. The details are spelt out in §6.6 of [9].

□

Proposition 1.3.2.9. *$U(n, K)$ is connected.*

Proof. The proof is essentially the same as that of Proposition 1.3.2.8, once we note that $U(n, K)$ is generated as an abstract group by matrices U_{ij} where $i < j$. □

1.4 Algebraic Group Actions

The most natural questions to ask about a group are always those around the way it acts on other objects. Here we begin to examine the way algebraic groups behave in their group actions.

1.4.1 Beginnings

Definition 1.4.1.1. A *G-variety* or *G-space* is a variety X along with an algebraic group G such that there exists a morphism of varieties

$$\begin{aligned} \alpha : G \times X &\longrightarrow X \\ (g, x) &\longmapsto g.x \end{aligned}$$

such that

1. $g.(h.x) = (gh).x$
2. $e.x = x$

for all $g, h \in G$, $x \in X$. We also say that G *acts (morphically)* on X.

Definition 1.4.1.2. We call a G-space X a *homogeneous space* for G if G acts transitively, that is if, for arbitrary $y \in X$, we have

$$G.y = X.$$

Definition 1.4.1.3. Let X be a G-space, and let $y \in X$. Define the *isotropy group* or *stabiliser* of y to be the group

$$G_y = \{g \in G \,|\, gy = y\}$$

Proposition 1.4.1.4. *The stabiliser G_y of an element $y \in X$ is indeed a group.*

Proof. Certainly G_y is nonempty, since $e \in G_y$. Suppose $g, h \in G_y$. Then

$$(gh)y = g(hy) = gy = y$$

so $gh \in G_y$. On the other hand,

$$h^{-1}y = h^{-1}hy = ey = y$$

and therefore G_y is a subgroup of G. □

Definition 1.4.1.5. Let X be a G-space with $Y, Z \subset X$. Define the *transporter* to be the set

$$\mathrm{Tran}_G(Y, Z) = \{x \in G \,|\, x.Y \subset Z\} \subset G$$

that is, all the elements of G which 'transport' Y to Z. Define the *centraliser* of Y in G to be the set

$$C_G(Y) = \bigcap_{y \in Y} G_y \subset G$$

that is, it is the set of elements of G which stabilise every element of the subset Y.

Note that there is some overlap in the various sets and groups we have defined, depending on the arguments. For example, $\mathrm{Tran}_G(y, y) = G_y$.

Definition 1.4.1.6. Let H be a subset of G. We denote by X^H the set of points of X which are fixed under the G-action by all the elements of H , that is

$$X^H = \{x \in X \,|\, h.x = x, \forall h \in H\}$$

If $g \in G$, then for ease of notation we write X^g for $X^{\{g\}}$.

Proposition 1.4.1.7. *Let X be a G-space, with $Y, Z \subset X$, such that Z is closed. Then*

1. $\mathrm{Tran}_G(Y, Z)$ *is a closed subset of G.*
2. *For each $y \in X$, G_y is a closed subgroup of G, and therefore so is $C_G(Y)$.*
3. *The fixed point X^g set of $g \in G$ is closed in X, and therefore so id X^G.*
4. *If G is connected, G stabilises each connected component of X, and hence acts trivially on X if X is finite.*

Proof. Let $y \in X$, and define the orbit map

$$\begin{aligned} \varphi_y : G &\longrightarrow X \\ g &\longmapsto g.y \end{aligned}$$

Then $\varphi_y = \varphi \circ (g \mapsto (g, y))$, where $\varphi : G \times X \to X$ is the morphism which defines the G-action. Being a composite of two morphisms, then, φ_y itself is a morphism. Now, for each $y \in Y$, we have $\varphi_y^{-1}(Z)$ is closed, since Z is. Moreover, clearly $\mathrm{Tran}_G(Y, Z) = \bigcap_{y \in Y} \varphi_y^{-1}(Z)$, so $\mathrm{Tran}_G(Y, Z)$ is closed, hence (1). For (2), simply note that $G_y = \mathrm{Tran}_G(\{y\}, \{y\})$, and since $\{y\}$ is closed, we can apply (1). Similarly, we have $C_G(Y) = \bigcap_{y \in Y} G_y$, and so $C_G(Y)$ is closed. For (3), first note that $X^G = \bigcap_{g \in G} X^g$, where X^g denotes the fixed point set of $g \in G$. It suffices to show, therefore, that for each $g \in G$, X^g is closed. Define a morphism

$$\begin{aligned} \psi : X &\longrightarrow X \times X \\ x &\longmapsto (x, g.x) \end{aligned}$$

Now $\Delta(X) \subset X \times X$ is closed, since X is a variety. Moreover, $X^g = \psi^{-1}\Delta(X)$, and so is closed, as required. For (4), first let $X = X_1 \cup \cdots \cup X_r$ be the decomposition of X into its irreducible components, which are of course each closed in X. Certainly, for each $g \in G$, the set $g(X_i)$ is connected, and so lies inside some X_j. We define $H = \mathrm{Tran}_G(X_i, X_i)$. Then H is evidently a subgroup, and its cosets are precisely the sets $\mathrm{Tran}_G(X_i, X_j)$. But there are only finitely many of these, since there are only finitely many connected components of X, and so H is of finite index in $G = G^\circ$. Moreover, H is closed by (1) above, and so Proposition 1.3.1.1(5) says $G \subset H$, and therefore $H = G$. That is, G stabilises each connected component of X. $\square$

Corollary 1.4.1.8. *Let G be an algebraic group, with $x \in G$ and H a closed subgroup. Then $Z(G), C_G(x), N_G(H)$ and $C_G(H)$ are each closed subgroups of G.*

Proof. G acts on itself by inner automorphisms, so we can apply Proposition 1.4.1.7(3) to give us $G^G = Z(G)$ is closed. If H is a closed subgroup, we can apply Proposition 1.4.1.7 (2) where the action is by inner automorphisms, to give us $C_G(H)$ and $C_G(x)$ closed. Similarly, $N_G(H) = \mathrm{Tran}_G(H, H)$. Indeed, if $x \in N_G(H)$, then $xHx^{-1} = H$, so certainly $x \in \mathrm{Tran}_G(H, H)$. For the reverse inclusion, suppose $x \in \mathrm{Tran}_G(H, H)$, that is $xHx^{-1} \subset H$. Let $h \in H$. Since H is a subgroup, $h^{-1} \in H$, and so $xh^{-1}x^{-1} = h'$ for some $h' \in H$. By taking inverses, we have $xhx^{-1} = h'^{-1} \in H$, whence $xHx^{-1} = H$, or $x \in N_G(H)$. Therefore $N_G(H) = \mathrm{Tran}_G(H, H)$, and so Proposition 1.4.1.7 (1) tells us that it is closed. $\square$

Definition 1.4.1.9. When G acts on two varieties X, Y, a morphism $\varphi : X \to Y$ is called *G-equivariant* if

$$\varphi(g.x) = g.\varphi(x)$$

for all $g \in G, x \in X$.

Example 9. Suppose $G \subset GL(V)$. Then G acts on itself and also on V, where V is viewed as the affine variety $K^n = \mathbf{A}^n$ for $n = \dim V$. Suppose $v \in V$, and consider the orbit map

$$\begin{aligned} \varphi : G &\longrightarrow V \\ g &\longmapsto g.v \end{aligned}$$

then φ is G-equivariant, since, for $x, g \in G$, we have

$$\varphi(gx) = (gx).v = g(xv) = g\varphi(x)$$

Example 10. Suppose V is a vector space. Then, as above, $GL(V)$ acts on V as a variety, and so acts on the set $V - \{0\}$ considered as an open affine subset of V. Furthermore, $GL(V)$ acts on $\mathbb{P}(V)$. The canonical map $V - \{0\} \to \mathbb{P}(V)$ is G-equivariant for any closed subgroup G of $GL(V)$.

1.4.2 G-modules

Definition 1.4.2.1. Suppose $\varphi : G \to GL(V)$ is a rational representation of an algebraic group G. If identify the vector space V with the affine variety $\mathbf{A}^n$, where $\dim V = n$, then we have an action of G on V as follows

$$\begin{aligned} G \times V &\longrightarrow V \\ (x, v) &\longrightarrow \varphi(x)(v) \end{aligned}$$

In this case, we call V a *G-module.*

Proposition 1.4.2.2. *Let φ, ψ be two representations $\varphi : G \to GL(V), \psi : G \to GL(W)$. Then G acts on the tensor product $V \otimes W$, and thus defines a rational representation $G \to GL(V \otimes W)$.*

Proof. Choose bases $\{v_1, \dots, v_n\}$ of V and $\{w_1, \dots, w_m\}$ of W. Then $\{v_i \otimes w_j \,|\, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes W$. Let $x \in G$, and define an operation

If we extend this linearly over $V \otimes W$, then this evidently defines a morphism of varieties $G \times (V \otimes W) \to V \otimes W$. $\square$

1.4.3 Deeper Results

Definition 1.4.3.1. A subset of a topological space is called *locally closed* if it is the intersection of a closed set and an open set.

Lemma 1.4.3.2. *A locally closed set is open in its closure.*

Proof. Let $U \cap F$ be an arbitrary locally closed set, where U is open and F is closed. But

$$U \cap F = U \cap \overline{U \cap F}$$

Indeed, if $x \in U \cap F$, then $x \in U \cap F \subset \overline{U \cap F}$ and $x \in U$, so $U \cap F \subset U \cap \overline{U \cap F}$. On the other hand, if $x \in U \cap \overline{U \cap F}$, then $x \in U$ and $x \in \overline{U \cap F}$. But $U \cap F \subset F$, and F is closed, so $\overline{U \cap F} \subset F$, which means $x \in F$. It follows that $U \cap F$ is open in its closure. $\square$

In general G-orbits are not closed subsets of X.

Proposition 1.4.3.3. *Suppose X is a G-space. Then each orbit is a locally closed subset of X whose boundary is a union of orbits of strictly lower dimension. In particular, orbits of minimal dimension are closed.*

Proof. Let $y \in X$ and let $Y = G.y$. Note that, by Lemma 1.3.2.5, $\bar{Y}$ is stable under the action of G. Moreover, since Y is the image of G under the morphism $g \mapsto g.y$, and so Y is constructible. Therefore, there is a subset $U \subset Y$ such that U is open and dense in $\bar{Y}$. Let $z \in U$, and since $Y = G.y$, we can denote this by $z = g.y$ for some $g \in G$. For any other element $z' \in Y$, we can similarly find $g' \in G$ such that $z' = g'y$, and so $z' = (g'g^{-1}).z$. It follows that $(g'g^{-1})U$ contains z' and is open in $\bar{Y}$. Therefore, each point of Y has a neighbourhood which is open in $\bar{Y}$, that is, Y is open in $\bar{Y}$.

We have, then, that $\bar{Y} - Y$ is closed in X and it is certainly of dimension lower than that of $\bar{Y}$. Note further that $\bar{Y} - Y$ is stable under the action of G, and so it is itself a union of G-orbits, as required. □

Recall, when G acts on a group N, the semi-direct product $N \rtimes G$ is the group with underlying set the cartesian product $N \times G$ and with multiplication defined by

$$(n_1, g_1)(n_2, g_2) = (n_1(g_1 \cdot n_2), g_1 g_2)$$

It is easy to see that

$$N \lhd N \rtimes G \quad \text{via} \quad n \mapsto (n, e)$$

and

$$G < N \rtimes G \quad \text{via} \quad g \mapsto (e, g)$$

So $z \in N \rtimes G$ can be written uniquely as $z = ng$. Consider the exact sequence

$$\begin{tikzcd} \{e\} \arrow[r] & N \arrow[r, "i"] & N \rtimes G \arrow[r, bend left, "\pi"] & G \arrow[l, bend left, "\sigma"] \arrow[r] & \{e\} \end{tikzcd}$$

Let G' be a group with subgroups G and N, with N being normal. Then G acts on N by inner automorphisms.

Proposition 1.4.3.4. *With notation as above, the map*

$$\begin{aligned} \alpha : N \rtimes G &\longrightarrow G' \\ (n, g) &\longmapsto ng \end{aligned}$$

is an isomorphism if and only if $G' = NG$ and $N \cap G = \{e\}$.

Proof. First of all we show that α is a group homomorphism. Indeed, suppose $(n_1, g_1), (n_2, g_2) \in N \rtimes G$. Then

$$\alpha((n_1, g_1)(n_2, g_2)) = \alpha((n_1(g_1 \cdot n_2), g_1 g_2)) = \alpha(n_1 g_1 n_2 g_1^{-1}, g_1 g_2) = n_1 g_1 n_2 g_2$$

as required. For the 'if' statement, we can now use the First Isomorphism Theorem for groups. Indeed, if $h \in G' = NG$, then $g = ng$ where $n \in N, g \in G$, and so $g = \alpha(n, g)$, which tells us that $\mathrm{im}(\alpha) = G'$. On the other hand, if $(n, g) \in \ker(\alpha)$, then $xy = e$, and so $x = y^{-1}$. Since both G and N are groups, they are closed under inversion, and so $x, y \in G \cap N$, which tells us that $(x, y) = (e, e)$, or $\ker \alpha = \{e\}$.

For the 'only if' statement, let $h \in G'$. Since α is assumed to be an isomorphism, there exists some element $(n, g) \in N \rtimes G$ such that $h = \alpha(n, g) = ng)$, and so $G' = NG$. On the other hand, suppose $h \in N \cap G$. Then $h \in G$, which in turn means $h^{-1} \in G$. We can therefore apply α to $(h, h^{-1}) \in N \rtimes G$, which gives us $\alpha(h, h^{-1}) = hh^{-1} = e$, and so since α is injective, this tells us $(h, h^{-1}) = (e, e)$, and so $N \cap G = \{e\}$, as required. □

Proposition 1.4.3.5. *If G and N are algebraic groups and G acts morphically as a group of automorphisms on N, then $N \rtimes G$ is an algebraic group.*

Proof. Let $(n_1, g_1), (n_2, g_2) \in N \rtimes G$. Then, by definition, their product is $(n_1(g_1 \cdot n_2), g_1 g_2)$. In terms of morphisms, the multiplication map $\mu : (N \rtimes G) \times (N \rtimes G) \to N \rtimes G$ is given by:

$$\mu((n_1, g_1), (n_2, g_2)) = (n_1(g_1 \cdot n_2), g_1 g_2) = (\mu_N(n_1, \rho(g_1, n_2)), \mu_G(g_1, g_2))$$

where μ_G is the multiplication morphism on G, μ_N is the multiplication morphism on N, and $\rho : G \times N \to N$ is the morphism defining the G-action on N. Therefore, begin a composite of morphisms of varieties, the multiplication map μ is itself a morphism of varieties.

On the other hand, if $(n, g) \in N \rtimes G$, then

$$\imath(n, g) = (\imath_N(n), \imath_G(g)) = (n^{-1}, g^{-1})$$

and so the inversion map $\imath : N \rtimes G \to N \rtimes G$ is also a morphism of varieties. Therefore $N \rtimes G$ is an algebraic group. $\square$

Example 11. The group $T(n, K)$ which consists of upper triangular, invertible matrices is the semidirect product of the the subgroup $D(n, K)$ and the normal subgroup $U(n, K)$. To prove this, we can simply apply Proposition 1.4.3.4, since it is clear that $T(n, K) = \big(U(n, K)\big)\big(D(n, K)\big)$, and that $D(n, K) \cap U(n, K) = \{e\}$.

1.4.4 Translation

Suppose G acts on an *affine* variety X (for example if $X = G$). Let $x \in G$, and consider the morphism

$$\begin{aligned} X &\longrightarrow X \\ y &\longmapsto x^{-1}.y \end{aligned}$$

Let $\tau_x : K[X] \to K[X]$ be the comorphism associated to this morphism. That is, if $f \in K[X]$, and $y \in X$, we have

$$(\tau_x f))(y) = f(x^{-1}.y)$$

Now define a map

$$\begin{aligned} \tau : G &\longrightarrow GL(K[X]) \\ x &\longmapsto \tau_x \end{aligned}$$

Lemma 1.4.4.1. *τ as defined above is a group homomorphism.*

Proof. Let $x, x' \in G$, $f \in K[X]$ and $y \in G$. We need to show that $\tau_{xx'} = \tau_x \circ \tau_{x'}$, so we test each function at f, which in turn gives us an element of $GL(K[X])$, which we test at y. That is,

$$\begin{aligned}\left(\tau_{xx'}(f)\right)(y) &= f\left((xx')^{-1}.y\right)\\ &= f(x'^{-1}x^{-1}.y)\\ &= (\tau_{x'}f)(x^{-1}.y)\\ &= \tau_x\left(\tau_{x'}f\right)(y)\end{aligned}$$

and so $\tau_{xx'} = \tau_x \circ \tau_{x'}$ as required. □

We call τ_x the *translation of functions* by x.

Proposition 1.4.4.2. *Suppose G acts on an affine variety X via the morphism $\varphi : G \times X \to X$. Suppose further that, for $f \in K[X]$, we have*

$$\varphi^*(f) = \sum f_i \otimes g_i, \quad f_i \in K[G], g_i \in K[X]$$

Then, for all $x \in G$, we have $\tau_x f = \sum f_i(x^{-1})g_i$.

Proof. Suppose firstly that $x \in G, y \in X$. Then, by a small abuse of notation, we have

$$\begin{aligned}f(x.y) &= f(\varphi(x,y))\\ &= \varphi^*(f)(x,y)\\ &= \left(\sum f_i \otimes g_i\right)(x,y)\\ &= \sum f_i(x)g_i(y)\end{aligned}$$

Now, since $(\tau_x f)(y) = f(x^{-1}.y)$ by definition, we have

$$(\tau_x f)(y) = \sum f_i(x^{-1})g_i(y)$$

But since y was arbitrary, this gives us $\tau_x f = \sum f_i(x^{-1})g_i$ as required. □

The following is both an example of a translation function and an important definition:

Definition 1.4.4.3. For $x \in G$ and with G acting on itself by left multiplication, that is, $x.y = xy$ for all $y \in G$, the associatated comorphism λ_x, defined to be

$$(\lambda_x f)(y) = f(x^{-1}y)$$

is called the *left translation of functions* by x. Similarly, for $x \in G$ and with G acting on itself by right inverse multiplication, that is, $x.y = yx^{-1}$ for all $y \in G$, the associatated comorphism ρ_x, defined to be

$$(\rho_x f)(y) = f(yx)$$

is called the *right translation of functions* by x.

We note some properties of these translation functions:

Proposition 1.4.4.4. *1. For $x \in G$, the morphisms λ_x and ρ_x commute.*

2. For $x, y \in G$ and $f \in K[G]$, we have $\lambda_{xy}(f) = \lambda_x(\lambda_y(f))$ and $\rho_{xy}(f) = \rho_y(\rho_x(f))$.

3. For $x \in G$ and $f, g \in K[G]$ we have $\lambda_x(fg) = (\lambda_x f)(\lambda_x g)$, and $\rho_x(fg) = (\rho_x f)(\rho_x g)$.

Proof. For (1), suppose $x, y \in G$ and $f \in K[G]$. Then, on the one hand

$$(\lambda_x \rho_x)(f)(y) = \lambda_x f(yx) = f(x^{-1}yx)$$

and on the other, we have

$$(\rho_x \lambda_x)(f)(y) = \rho_x f(x^{-1}y) = f(x^{-1}yx)$$

which tells us that $\rho_x \lambda_x = \lambda_x \rho_x$.

(2) simply follows as an application of Lemma 1.4.4.1, and (3) is by the definition of multiplication in $K[G]$. □

Of course, Proposition 1.4.4.4 (2) says that the maps $\lambda, \rho : G \to GL(K[G])$ are group homomorphisms.

Lemma 1.4.4.5. *Let H be a closed subgroup of an algebraic group G, and I the ideal of $K[G]$ which vanishes on H, that is, $I = \mathscr{I}(H)$. Then*

$$H = \{x \in G \mid \rho_x(I) \subset I\}$$

Proof. First we show $H \subset \{x \in G \mid \rho_x(I) \subset I\}$, so suppose $x \in H$. We need to show that $\rho_x(I) \subset I$, that is, for all $f \in I$, we need $(\rho_x f)(y) = 0$ for all $y \in H$. But $(\rho_x f)(y) = f(yx) = 0$ since $yx \in H$, and so $x \in \{x \in G \mid \rho_x(I) \subset I\}$.

For the reverse direction, suppose $x \in G$ is such that $\rho_x(I) \subset I$. In particular, for all $f \in I$, we have $0 = \rho_x(f)(e) = f(x)$. That is, for all $f \in I$, we have $f(x) = 0$, and so $x \in \mathscr{V}(I) = \mathscr{V}(\mathscr{I}(H))$. But since H is a closed subset of an affine variety G, H is an algebraic set, and so $\mathscr{V}(\mathscr{I}(H)) = H$, which tells us that $\{x \in G \mid \rho_x(I) \subset I\} \subset H$, as required. □

Corollary 1.4.4.6. *Let H be a closed subgroup of an algebraic group G, and I the ideal of $K[G]$ which vanishes on H, that is, $I = \mathscr{I}(H)$. Then*

$$H = \{x \in G \mid \rho_x(I) = I\}$$

Proof. By Lemma 1.4.4.5, we know

$$H = \{x \in G \mid \rho_x(I) \subset I\}$$

But since H is a subgroup, for any $x \in H$, we also have $x^{-1} \in H$. This in turn gives us

$$\rho_{x^{-1}}(I) \subset I. \tag{1.7}$$

By Proposition 1.4.4.4(2) we know that $\rho_x \rho_{x^{-1}} I = I$, so applying ρ_x to both sides of equation (1.7) gives

$$I = \rho_x \left(\rho_{x^{-1}}(I)\right) \subset \rho_x(I)$$

and so $H \subset \{x \in G \mid \rho_x(I) = I\}$. The reverse inclusion is simple, since clearly

$$\{x \in G \mid \rho_x(I) = I\} \subset \{x \in G \mid \rho_x(I) \subset I\} = H$$

where the last equality is by Lemma 1.4.4.5. □

Proposition 1.4.4.7. *Let an algebraic group G act on an affine variety X, where $\varphi : G \times X \to X$ is the morphism defining the G-action on X, and let F be a finite dimensional subspace of $K[X]$. Then*

1. *there is a finite dimensional subspace $E \subset K[X]$ with $F \subset E$ which is stable under translations τ_x, for all $x \in G$.*

2. *F itself is stable under τ_x for all $x \in G$, if and only if $\varphi^* F \subset K[G] \otimes_K F$*

Proof. For (1) we can use an induction argument on the dimension of F. First suppose that F is the K-span of a single $f \in K[X]$. We use the familiar trick of writing $\varphi^* f = \sum f_i \otimes g_i$, noting that of course $1 \leq i \leq r$ for some r. Now, by Proposition 1.4.4.2, for any $x \in G$, we have $\tau_x f = \sum f_i(x^{-1}) g_i$, which is to say that $\tau_x f$ is a linear sum of the finitely many g_i's. These g_i's span a finite dimensional space, call it V. If we therefore define E to be the span of all the $\tau_x f$, for $x \in G$, then $E \subset V$ and so E is finite dimensional, and certainly contains $f = \tau_e f$. Moreover, given $x, y \in G$, we have $\tau_x(\tau_y f) = \tau_{xy}(f) \subset E$, where the equality is by Lemma 1.4.4.1, and so E is stable under all translations τ_x, as required.

Now suppose that $\dim F = n$. In this case we simply need to consider a direct sum $F = F_1 \oplus F_2$, where F_1, F_2 are of strictly lesser dimension than F, to define sets E_1, E_2 which satisfy our criteria by the induction hypothesis. Then certainly $E = E_1 \oplus E_2$ contains F and is stable under all translations τ_x.

For (2), if $\varphi^* F \subset K[G] \otimes F$, then we can take the g_i from (1) to lie in F, and so the E we construct will also lie in F, that is, F will be stable under all translations τ_x. For the reverse direction, suppose $\{f_1, \ldots, f_m\}$ is a basis for F. Note that, unlike in (1), these f_i's now lie in $K[X]$, and not in $K[G]$. We now extend this basis of F to a basis $\{f_i\} \cup \{h_j\}$ of $K[X]$, the h_j not necessarily finite. If, for arbitrary $f \in K[X]$, we now write

$$\varphi^* f = \sum r_i \otimes f_i + \sum s_j \otimes h_j \tag{1.8}$$

where $r_i, s_j \in K[G]$, we can apply Proposition 1.4.4.2 again to get

$$\tau_x f = \sum r_i(x^{-1}) f_i + \sum s_j(x^{-1}) h_j \subset F$$

This means that $s_j(x^{-1}) = 0$ for all $x \in G$, and therefore $s_j = 0$. Substituting this information back into equation (1.8), we see that $\varphi^* F \subset K[G] \otimes F$, as required. □

Theorem 1.4.4.8. *Let G be an algebraic group. Then G is isomorphic to a closed subgroup $GL(n, K)$ for some n.*

Proof. Let $f_1, \ldots, f_n$ be generators of the affine algebra $K[G]$. If we define F to be the K-vector space spanned by these f_i, Proposition 1.4.4.7 tells us that there is a finite dimensional space E with $F \subset E$ and such that $\rho_x E \subset E$, for all $x \in G$. If we change notation we can assume $f_1, \ldots, f_n$ form a basis for this E, and that they still generate $K[G]$.

Suppose now that $\varphi : G \times X \to X$ defines the G-action which we use to define ρ_x, namely the action of right multiplication by x^{-1}. Then Proposition 1.4.4.7 (2) says that $\varphi^* E \subset K[G] \otimes E$. In particular, for each of the basis elements f_i, we can write

$$\varphi^* f_i = \sum_{1 \leq j \leq n} m_{ij} \otimes f_j$$

for some $m_{ij} \in K[G]$. We use Lemma 1.4.4.1 again to get

$$\rho_x f_i = \sum m_{ij}(x) f_j$$

This simply says that the matrix of $\rho_x|_E$ relative to the basis $\{f_1, \ldots, f_n\}$ is $(m_{ij}(x))$. We can therefore define a morphism of algebraic groups

$$\begin{aligned} \psi : G &\longrightarrow GL(n, K) \\ x &\longmapsto (m_{ij}(x)) \end{aligned}$$

Define $G' = \mathrm{im}\psi$. We know from Proposition 1.3.2.1 (2) that G' is closed in $GL(n, K)$, so it only remains to show that ψ is an isomorphism onto G', that is, we need to show that ψ is injective. Note that it suffices to show that $\psi^* : K[G'] \to K[G]$ is surjective. We do this by first considering the equation

$$f_i(x) = f_i(ex) = \rho_x f_i(e) = \sum m_{ij}(x) f_j(e)$$

which tells us $f_i = \sum f_j(e) m_{ij}$, and therefore the m_{ij} generate $K[G]$, since the f_i do. Recall that the m_{ij} are elements of the matrices (m_{ij}), and so it is easy to see that the coordinate functions T_{ij} of $GL(n, K)$ restricted to G' are sent to the function m_{ij}. This tells us that ψ^* is surjective, and the result is proved. □

1.5 Quotients of Algebraic Groups

We conclude this introductory chapter with a description of how to form quotients of algebraic groups. In fact, for the most part, these quotients will not be groups at all, but rather will be homogeneous G-spaces which are quasi-projective. This space does become a linear algebraic group, however, if the group to be quotiented out is normal. We should also note that the entire thrust behind the theory of this section lies in Chevalley's Theorem, Theorem 1.5.1.2. The technique set out in this theorem is useful in later applications, too, and so it forms a central plank in the general theory of linear algebraic groups.

1.5.1 Preliminary Results

The following results are required to construct quotients. Note at the outset that $\mathfrak{gl}(n, K)$ denotes the Lie algebra of $GL(n, K)$, which consists of all $n \times n$ matrices ([6], pg 68). Now $GL(n, K)$ and $\mathfrak{gl}(n, K)$ act on exterior powers of the vector space $W = K^n$ in the obvious way.

Lemma 1.5.1.1. *Let $x \in GL(W)$ and $\mathsf{x} \in \mathfrak{gl}(W)$, and let $M \subset W$ be a d-dimensional subspace. Further, define*

$$L = \wedge^d M$$
$$V = \wedge^d W$$

Note that L is one-dimensional, and that $L \subset V$. Then

1. $(\wedge^d x)L = L \Longleftrightarrow x(M) = M$
2. $(d\wedge^d)(\mathsf{x})L \subset L \Longleftrightarrow \mathsf{x}(M) \subset M$

Proof. For (1), We first choose a basis $w_1, \dots, w_n$ of W which satisfies the following two conditions: firstly, $w_1, \dots, w_d$ should span M, and secondly, $w_{l+1}, \dots, w_{l+d}$ should span $x(M)$, for some $l \geq 0$.

On the one hand, if $x(M) = M$ then $l = 0$, and so

$$\begin{aligned}(\wedge^d x)(w_1 \wedge \cdots \wedge w_d) &= x(w_1) \wedge \cdots \wedge x(w_d)\\ &= \mu(w_1 \wedge \cdots \wedge w_d) \in L\end{aligned}$$

for some scalar μ, as required. On the other hand, suppose $(\wedge^d x)L = L$. We aim to show that $l = 0$. It is clear by the definition of L that $w_1 \wedge \cdots \wedge w_d$ spans L, and so by our hypothesis,

$$(\wedge^d x)(w_1 \wedge \cdots \wedge w_d) = \lambda(w_1 \wedge \cdots \wedge w_d)$$

for some nonzero scalar λ. On the other hand,

$$\begin{aligned}(\wedge^d x)(w_1 \wedge \cdots \wedge w_d) &= x(w_1) \wedge \cdots \wedge x(w_d)\\ &= \mu(w_{l+1} \wedge \cdots \wedge w_{l+d})\end{aligned}$$

for some scalar μ, and so these two relations force $l = 0$, as required.

For (2), again, it is immediate that if $\mathsf{x}(M) \subset M$, then $(d\wedge^d)(\mathsf{x})L \subset L$. Of course, equality does not hold, since the right hand side might be equal to 0. For the reverse direction, note firstly that we may assume $M \cap \mathsf{x}(M) = \{0\}$. Otherwise, choose a basis $\{w_1, \dots, w_l, \dots, w_d\}$ of M such that

$$\begin{aligned}\mathsf{x}(w_i) &\in M \qquad 1 \leq i \leq l\\ \mathsf{x}(w_i) &\notin M \qquad l+1 \leq i \leq d\end{aligned}$$

We now choose an element $\mathsf{y} \in \mathfrak{gl}(W)$ such that

$$\begin{aligned}\mathsf{x}(w_i) &= \mathsf{x}(w_i) \qquad 1 \leq i \leq l\\ \mathsf{y}(w_i) &= 0 \qquad l+1 \leq i \leq d\end{aligned}$$

and such that y sends everything outside M to zero, too. Clearly $\mathsf{y}(M) \subset M$, and so $(d\wedge^d)(\mathsf{y})L \subset L$. Set $\mathsf{w} = \mathsf{x} - \mathsf{y}$. Certainly $(d\wedge^d)(\mathsf{w})L \subset L$, and, moreover, $M \cap \mathsf{w}M = \{0\}$. Note that if w stabilises M, then x does, too, so we can replace x with w for the remainder of the proof. Note that now our basis $\{w_1, \dots, w_l, \dots, w_d\}$ of M is such that so as to ensure that $\mathsf{x}(M) = \operatorname{span}\{\mathsf{x}(w_{l+1}), \dots, \mathsf{x}(w_d)\}$ and $\mathsf{x}(w_i) = 0$ for all $1 \leq i \leq l$.

Now,

$$(d\wedge^d)(\mathsf{x})(w_1 \wedge \cdots \wedge w_d) = \sum_i w_1 \cdots \wedge \mathsf{x}(w_i) \wedge \cdots \wedge w_d$$

But by assumption, this left hand side is a multiple of $w_1 \wedge \cdots \wedge w_d$, which by our construction is possible only if this multiple is 0. That is, $\mathsf{x}(w_i) = 0$ for all i, and so $\mathsf{x}(M) = 0 \subset M$, as required. $\square$

Theorem 1.5.1.2 (Chevalley). *Let G be an algebraic group, H a closed subgroup. Then there is a rational representation $\varphi : G \to GL(V)$ and a one-dimensional subspace L of V such that*

1. $H = \{x \in G \mid \varphi(x)L = L\}$, *and*
2. $\mathfrak{h} = \{\mathsf{x} \in \mathfrak{g} \mid d\varphi(\mathsf{x})L \subset L\}$.

Proof. We start by proving (1). Let $I = \mathscr{I}(H)$, the ideal of $K[G]$ which vanishes on H. Recall, since $K[G]$ is noetherian, I is finitely generated, by elements $f_1, \dots f_m$, say. Define a finite dimensional K-vector subspace $F = \operatorname{span}\{f_1, \dots, f_m\}$. Then, by Proposition 1.4.4.7(1), there exists a finite dimensional vector space W such that $F \subset W \subset K[G]$, and $\rho_x(W) \subset W$ for all $x \in G$. We define another vector space, $M = W \cap I$. Note that, in particular, $f_1, \dots, f_m \in M$, and so M again generates I. We claim that $\rho_x(M) \subset M$ for all $x \in H$. Indeed, Corollary 1.4.4.6 tells us $H = \{x \in G \mid \rho_x(I) = I\}$, and by definition W is stable under all translations ρ_x with $x \in G$, so is certainly stable under those with $x \in H$. Note that we can use the same trick we used in the proof of Lemma 1.4.4.6 to show that, since $\rho_{x^{-1}}(M) \subset M$ for all $x \in H$, we also have $M \subset \rho_x(M)$, and therefore $\rho_x(M) = M$ for all $x \in H$.

We can go one step further, and claim that $H = \{x \in G \mid \rho_x(M) = M\}$. We have already shown that $H \subset \{x \in G \mid \rho_x(M) = M\}$, so for the reverse inclusion, suppose that $x \in G$ is such that $\rho_x(M) = M$. Since I is generated by M, which is the same as saying $I = MA$ where $A = K[G]$, we see that

$$\rho_x(I) = \rho_x(MA) = \rho_x(M)\rho_x(A) = MA = I$$

where the second equality is by Proposition 1.4.4.4(3). But, appealing to Corollary 1.4.4.6 again tells us that $x \in H$, and so $\{x \in G \mid \rho_x(M) = M\} \subset H$.

To summarise, so far we have $H = \{x \in G \mid \rho_x(M) = M\}$, which is not quite what we want to prove. To get to where we're going, we define $V = \wedge^d W, L = \wedge^d M$, where $d = \dim M$. Moreover, we define a representation

$$\begin{aligned} \varphi : G &\longrightarrow GL(V) \\ x &\longmapsto (\wedge^d \rho_x) \end{aligned}$$

Then, for $x \in H$, since $\rho_x(M) = M$, Lemma 1.5.1.1(1) tells us that $\varphi(x)L = L$, so $H \subset \{x \in G \,|\, \varphi(x)L = L\}$. On the other hand, if $\varphi(x)L = L$, Lemma 1.5.1.1(1) says $\rho_x(M) = M$, and so $x \in H$, which gives us $\{x \in G \,|\, \varphi(x)L = L\} \subset H$, and so the result (1) is proven. The proof of (2) is similar, using Lemma 1.5.1.1(2) and also a Lie algebra analogue of Proposition 1.4.4.4, given as Lemma 9.4 in [6]. $\square$

Definition 1.5.1.3. Let G be an algebraic group. A *character* of G is a morphism of algebraic groups

$$\chi : G \longrightarrow \mathbb{G}_m.$$

We denote the set of characters of G by $\mathrm{X}(G)$.

Lemma 1.5.1.4. *Let $\psi : G \to GL(V)$ be a rational representation. Then the set*

$$\{V_\chi \,|\, \chi \in X(G)\}$$

is linearly independent. In particular, only finitely many of the spaces V_χ are nonzero.

Proof. Suppose the contrary, and choose a minimal $n \geq 2$ and vectors $v_i \in V_{\chi_i}$ such that $v_1 + \cdots + v_n = 0$. Since χ_1, χ_2 are distinct, there exists an element $x \in G$ such that $\chi_1(x) \neq \chi_2(x)$. Take this x, and consider the equation

$$0 = \varphi(x)0 = \varphi(x) \sum_{1 \leq i \leq n} v_i = \sum_{1 \leq i \leq n} \chi_i(x) v_i \tag{1.9}$$

If we multiply both sides of (1.9) by the nonzero element $\chi_1(x)^{-1}$, we get

$$0 = \sum_{1 \leq i \leq n} \chi_1(x)^{-1}\chi_i(x) v_i = v_1 + \sum_{2 \leq i \leq n} \chi_1(x)^{-1}\chi_i(x) v_i \tag{1.10}$$

Now we subtract the equation $0 = \sum v_i$ from equation (1.10), to get

$$0 = \sum_{2 \leq i \leq n} \big(\chi_1(x)^{-1}\chi_i(x) - 1\big) v_i \tag{1.11}$$

Since $\chi_1(x)^{-1} \neq \chi_2^{-1}$, we know that the coefficient of v_2 in equation (1.11) is not zero. Therefore equation (1.11) is a nontrivial dependence on at most $n-1$ different terms, contradicting the minimality of n. $\square$

Lemma 1.5.1.5. *Suppose N is a closed, normal subgroup of an algebraic group G, with $\varphi : G \to GL(V)$ a rational representation of G. Then $\varphi(G)$ permutes the characters of N, and therefore G acts naturally on $\mathrm{X}(N)$.*

Proof. We can, without loss of generality, simply assume that $G \subset GL(V)$, for convenience. Consider a character $\chi \in \mathrm{X}(G)$, and fix an element $x \in G$. We define another character of N as follows:

$$\begin{aligned} \chi' : N &\longrightarrow \mathbb{G}_m \\ y &\longmapsto \chi(x^{-1}yx) \end{aligned}$$

We then claim that $x.V_\chi \subset V_{\chi'}$. Indeed, suppose $v \in V_\chi$. Then, for any $y \in N$, we have

$$y(x.v) = x(x^{-1}yx).v = x.\chi(x^{-1}yx)v = \chi(x^{-1}yx)x.v = \chi'(y)(x.v) \qquad (1.12)$$

as required. This is sufficient to show that the elements of G act on $\mathrm{X}(G)$, and, indeed, we could write $\chi' = x.\chi$. $\square$

1.5.2 A Homogeneous Space

As set out in §1.6 of [6], projective n-space, $\mathbb{P}^n$, is given the structure of a variety by equipping it with a topology where closed sets are the common zeros of a collection of homogeneous polynomials in $n+1$ indeterminates. A *projective variety* is a closed subset of $\mathbb{P}^n$. In the terminology of [6], a *quasi-projective variety* is an open subset of a projective variety.

Theorem 1.5.2.1. *Let H be a closed subgroup of G. There exists a quasi-projective, homogeneous G-space Y, and a point $y \in Y$, such that*

1. *The isotropy group of y in G is equal to H,*
2. *The orbit morphism $\pi : g \mapsto g.y$ of G onto Y is such that the fibres of π are the cosets gH, for $g \in G$,*
3. *The linear map $d\pi_e : \mathfrak{g} \to \mathscr{T}(Y)_y$ is surjective.*

Proof. We use Theorem 1.5.1.2 to construct a representation $\varphi : G \to GL(V)$ and a line $L = Kv$ for some $v \in V$ such that

$$H = \{g \in G \,|\, \varphi(g)L = L\} \qquad (1.13)$$

and

$$\mathfrak{h} = \{\mathrm{y} \in \mathfrak{g} \,|\, d\varphi(\mathrm{y})L \subset L\} \qquad (1.14)$$

Let $y = [v] \in \mathbb{P}(V)$ be the point in $\mathbb{P}(V)$ corresponding to L.

It is easy to define a G-action on $\mathbb{P}(V)$ by sending elements $g \in G$ to $[\varphi(g)v] \in \mathbb{P}(V)$. Call Y the G-orbit of this action. That is,

$$Y = \{[\varphi(g)v] \,|\, g \in G\}$$

and we set $\pi : G \to Y$ to be the orbit map. By obviously, Y is a homogeneous G-space, and, moreoever Proposition 1.4.3.3 tells us that Y is locally closed in $\mathbb{P}(V)$, and is therefore a quasi-projective variety, by definition.

Suppose $g \in G_y$, the isotropy group of y. Then $g.y = [\rho(g)v] = [v]$, which is to say $\rho(g)v = kv$ for some scalar k. But (1.13) then says that $g \in H$. On the other hand, suppose $g \in H$. Then $\rho(g)L = L$ and so $g.y = [\rho(g)L] = [L] = y$, and so $g \in G_y$. Therefore $G_y = H$, proving (1).

Suppose now that $y = [\rho(g)v] \in Y$. We claim that $\pi^{-1}(y) = gH$. Indeed, let $h \in H$. Then

$$\pi(gh) = (gh).y = [\rho(g)\rho(h)v] = [\rho(g)v]$$

where the last equality uses (1.13) again. On the other hand, suppose $g' \in \pi^{-1}(y)$, so $g'.y = g.y$, and therefore $g^{-1}g'.y = y$. By (1) above, this means $g^{-1}g' \in H$, and so $g' = g(g^{-1}g') \in gH$. Therefore $\varphi^{-1}(y) = gH$, proving (1).

We now move to (3), and so we need to calculate $d\pi_e$. If we define the map $\sigma : V - \{0\} \to \mathbb{P}(V)$ in the canonical way, then $d\sigma_v$ is a surjective, linear map from V to $\mathscr{T}(\mathbb{P}(V))_{[v]}$, which has kernel L. Consider now the elementary morphism $\omega : GL(V) \to V$ which sends f to $f(v)$. Then since this map is linear, it follows that $d\omega_e$ sends an element $\mathsf{f} \in \mathfrak{gl}(V)$ to $\mathsf{f}(v)$.

Note now that $\pi(g) = [\rho(g)v] = \sigma \circ \omega \circ \varphi(g)$. Therefore

$$d\pi_e(\mathsf{y}) = d\sigma_v \circ d\omega_e \circ d\varphi_e(\mathsf{y}) = d\sigma_v(d\varphi_e(\mathsf{y})v)$$

So $d\pi(\mathsf{y})$ is the image of $d\varphi_e(\mathsf{y})v$ in $\mathscr{T}(\mathbb{P}(V))_{[v]}$. Therefore if $\mathsf{y} \in \ker(d\pi)$, then $d\varphi(\mathsf{y})v \in L$, and so by (1.14), $\ker(d\pi) \subset \mathfrak{h}$. On the other hand, if $\mathsf{y} \in \mathfrak{h}$, then $d\varphi(\mathsf{y})v \in L = \ker d\sigma_v$, and so $d\pi_e(\mathsf{y}) = d\sigma_v(d\varphi_e(\mathsf{y})v) = 0$. Therefore $\mathsf{y} \in \ker(d\pi)$, and so we have $\ker(d\pi) = \mathfrak{h}$.

By another application of Theorem 4.3 of [6], $\dim G = \dim G_y + \dim G.y = \dim H + \dim Y$, we have $\dim Y = \dim \mathfrak{g} - \dim \mathfrak{h} = \dim \mathfrak{g} - \dim \ker d\pi$. But since $d\pi$ is a linear map, this latter equality becomes $\dim Y = \dim \mathrm{im} d\pi$, and so $d\pi$ is surjective, as required. □

1.5.3 Quotients

Definition 1.5.3.1. Let H be a closed subgroup of an algebraic group G. A *quotient* of G by H is a pair (X, x) consisting of a variety X on which G acts transitively, and a point $x \in X$ such that $H = G_x$. Furthermore, this pair must satisfy the following universal property:

- If (X', x') is a pair consisting of a variety X' on which G-acts transitively, and a point $x' \in X'$ such that $H \subset G_{x'}$, then there is a unique G-morphism $\phi : X \to X'$ such that $\phi(x) = x'$.

Theorem 1.5.3.2. *Let H be a closed subgroup of an algebraic group G. Then (Y, y) as identified above in Theorem 1.5.2.1 is a quotient of G by H, and this is unique up to isomorphism.*

Proof. We first prove uniqueness, so suppose (X, x) and (X', x') are both quotients of G by H. Then, by definition, we have unique morphisms $\phi : X \to X'$ with $\phi(x) = x'$ and $\psi : X' \to X$ with $\psi(x') = x$. We can compose these to give a morphism $\psi \circ \phi$ from X to X such that $\psi \circ \phi(x) = \psi(x') = x$. But the identity is also a morphism from X to X which sends x to x, and since (X, x) is a quotient, these morphisms must coincide, that is, $\psi \circ \phi = \mathrm{id}_X$. Similarly, $\phi \circ \psi = \mathrm{id}_{X'}$ and so ϕ is bijective, with $\psi = \phi^{-1}$, which itself means ϕ^{-1} is a morphisms of varieties. Therefore ϕ is an isomorphism of varieties, or $X \cong X'$, as required.

We now go on to prove existence. Let $X = G/H$ be the set of cosets gH for $g \in G$ with canonical map $q : G \to G/H$, and set $x = eH = H$. We wish to define X as a ringed space, that is, a topological space equipped with a sheaf of rings $\mathcal{O}$. For the topology, we will declare a set $U \subset G/H$ to be open if $q^{-1}(U)$ is open in G, so q is continuous. Indeed, q is an open map. To see this, let $W \subset G$ be open. We need to show that $q(W)$ is open in G/H, which is to say $q^{-1}q(W)$ is open in G, so let $g \in q^{-1}q(W)$. Then $q(g) = q(w)$ for some $w \in W$, and so $w^{-1}g \in H$. Since any right translation is an isomorphism of algebraic groups, the set $W' = Ww^{-1}g$ is open in G. But $g \in W'$, and $W' \subset q^{-1}q(W)$, which means that this latter set is open in G, and so $q(W)$ is open in G/H.

Now suppose U is an open subset of G/H. Define the ring $\mathcal{O}(U)$ as follows:

$$\mathcal{O}(U) = \{f : U \to K \mid f \circ q \in \mathcal{O}_G\left(q^{-1}(U)\right)\}$$

It is not difficult to show that the sheaf conditions hold for this construction, so that $(G/H, \mathcal{O})$ is indeed a ringed space. Now certainly G acts transitively on G/H as a set, by $g'.gH = (g'g)H$, and indeed defines a transitive G action on G/H as a ringed space.

We also want to check the universal property holds for G/H, so suppose (X', x') is a pair as in Definition 1.5.3.1. Then the map

$$\phi : G/H \longrightarrow X' \tag{1.15}$$

$$gH \longmapsto g.x' \tag{1.16}$$

is a well defined, since $G_x = H \subset G_{x'}$, and, indeed, ϕ is a morphism of ringed spaces.

We have a pair $(X, x) = (G/H, H)$ which satisfies all of the criteria of Definition 1.5.3.1 except that we have not yet shown that G/H is a variety. To do so, we will take the variety Y of Theorem 1.5.2.1 and define a G-equivariant morphism of ringed spaces which sends G/H to Y, and in particular sends the point H to y. This will suffice to give G/H the structure of a variety. The morphism to use is $F : G/H \to Y$, which sends gH to $g.y$. Note that $\pi = F \circ q$ where π is the G-orbit map of y, as in Theorem 1.5.2.1.

Now F is bijective by Theorem 1.5.2.1(2). Moreover, F is continuous. Indeed, if $U \subset Y$ is open, we need to show $F^{-1}(U)$ is open in G/H. But this is the case if and only if $q^{-1}F^{-1}(U)$ is open in G, but this set is equal to $\pi^{-1}(U)$ which is open since π is continuous. Furthermore, F^{-1} is continuous, too, since π is an open map. Indeed, this last fact is a consequence of Theorem 4.3.3. in [9]. This theorem states that, given a G-equivariant morphism of homogeneous spaces $\nu : Z_1 \to Z_2$, then given another a third varitety Z_3, the induced map

$$\nu \times \mathrm{id} : Z_1 \times Z_3 \to Z_2 \times Z_3$$

is open. Evidently this theorem is immediately applicable to our map π. Therefore, F is a homeomorphism.

It is a little more complicated to show that, for each open set $U \subset G/H$, the ring homomorphism

$$F_U : \mathcal{O}(U) \longrightarrow \mathcal{O}_Y(F^{-1}(U))$$
$$f \longrightarrow f \circ F$$

is an isomorphism of rings. The proof requires particular results about morphisms which have been omitted from this paper, but the essence of the argument relies on the fact, given in Theorem 1.5.2.1(3), that the differential of the quotient morphism $d\pi$ is surjective, which in turn means that the morphism π is *separable.* Separable morphisms are sufficiently 'well behaved' that the above ring homomorphisms are indeed isomorphisms. The details can be found in [9], pages 134-135. □

To summarise,

Corollary 1.5.3.3. *If H is a closed subgroup of G, then G/H is a quasi-projective variety, is irreducible if G is connected, and* $\dim G/H = \dim G - \dim H$.

Proof. The first part follows immediately from Theorem 1.5.3.2, while the statement about dimension is derived from Theorem 1.5.2.1(1) along with yet another application of Theorem 4.3.3 of [6], which, when applied to orbit maps, says $\dim G = \dim G_x + \dim G.x$. Since, in this context, $G.x = H$ and $G.x = G/H$, the result follows. □

Example 12. If $G = GL(n, K)$ and $H = T(n, K)$, then G/H consists of the flags of $V = K^n$. This is called the *flag variety*, and we denote it by $\mathscr{F}(V)$. It is in fact a closed subset of projective space, and so is projective. The way to show this is by first giving building a *Grassman variety* $\mathfrak{G}_d(V)$, which consists of all the d-dimensional subspaces of V. Indeed, it is straightforward enough to construct an injective map $f : \mathfrak{G}_d(V) \to \mathbb{P}(\wedge^d V)$ by sending a subspace D to the point in $\mathbb{P}(\wedge^d V)$ corresponding to $\wedge^d D$. In fact, the image of f turns out to be closed, and so $\mathfrak{G}_d(V)$ is projective. One then in turn shows that $\mathscr{F}(V)$ is closed in the product $\mathfrak{G}_1(V) \times \cdots \times \mathfrak{G}_n(V)$. The details can be found in §1.8 of [6].

Example 13. If $G = SL(2, K), H = T(2, K)$, then $G/H = \mathbb{P}^1$. In fact, this is a special case of a more general theorem, which we prove as Theorem 3.3.3.2.

Proposition 1.5.3.4. *Let N be a closed, normal subgroup of an algebraic group G. Then*

1. *G/N, endowed with the usual group structure, is a linear algebraic group,*
2. *G/N is affine,*
3. $\mathscr{L}(G/N) = \mathfrak{g}/\mathfrak{n}$.

The proof of Proposition 1.5.3.4 requires the following lemma:

Lemma 1.5.3.5. *Let H_1 be a closed subgroup of an algebraic group G_1 and H_2 a closed subgroup of an algebraic group G_2. Then there is an isomorphism of homogeneous $G_1 \times G_2$-spaces*

$$(G_1 \times G_2)/(H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2$$

Proof. This follows from the uniqueness of quotients. Indeed, the variety $G_1/H_1 \times G_2/H_2$ is such that the elements of $(G_1 \times G_2)$ which stabilise the point $H_1 \times H_2 \in (G_1 \times G_2)/(H_1 \times H_2)$ are precisely those which stabilise the point $(H_1, H_2) \in G_1/H_1 \times G_2/H_2$. The universal property of Definition 1.5.3.1 therefore is satisfied by $G_1/H_1 \times G_2/H_2$ since it is satisfied by $(G_1 \times G_2)/(H_1 \times H_2)$, and so the uniqueness proven in Theorem 1.5.3.2 ensures that these two varieties are isomorphic. $\square$

Proof of Proposition 1.5.3.4. To prove (1), that G/N is an algebraic group, we need to show that the inversion map and the multiplication map are morphisms of algebraic groups.

Define a $(G \times G)$-action on G/N as follows:

$$\begin{aligned}(G \times G) \times G/N &\longrightarrow G/N \\ \big((g_1, g_2), xN\big) &\longmapsto g_1 x g_2^{-1} N\end{aligned}$$

This action is clearly transitive. Moreover, if $(n_1, n_2) \in N \times N$, then

$$\big((n_1, n_2), N\big) = n_1 n_2^{-1} N = N$$

since N is normal. Therefore, the isotropy group of the point $N \in G/N$ contains $N \times N$. By the universal property given in Definition 1.5.3.1, combined with Lemma 1.5.3.5, there is a unique morphism

$$\begin{aligned}\phi : G/N \times G/N &\longrightarrow G/N \\ (g_1 N, g_2 N) &\longmapsto g_1 g_2^{-1} N\end{aligned}$$

If we restrict ϕ to the closed subset $G/N = \{N\} \times G/N$, we get a morphism

$$\begin{aligned}\imath : G/N &\longrightarrow G/N \\ g_2 N &\longmapsto g_2^{-1} N\end{aligned}$$

which is precisely the inversion map on the group G/N. Moreover, the composition μ, defined

$$G/N \times G/N \xrightarrow{\mathrm{id} \times \imath} G/N \times G/N \xrightarrow{\phi} G/N$$

sends $(g_1 N, g_2 N)$ to $(g_1 g_2 N)$, which is precisely the product map for the group G/N. Therefore the inversion map and product map are both morphisms of varieties, and so G/N is an algebraic group, proving (1).

We move on to the proof of (2). As usual, we use Theorem 1.5.1.2 to define a representation $\varphi : G \to GL(V)$ and a line $L \subset V$ such that $N = \{x \in G \,|\, \varphi(x)L = L\}$ and $\mathscr{L}(N) = \{\mathrm{x} \in \mathrm{x} \,|\, d\varphi(\mathrm{x})L \subset L\}$. Since L is stabilised by N, it constitutes a one dimensional representation of N, and so we can construe it as the weight space of a character $\chi_0 \in \mathrm{X}(N)$. That is, if $v \in L$, then $\varphi(x)v = \chi_0(x)v$ for all $x \in N$.

We consider the subspace $V' \subset V$ defined as follows:

$$V' = \bigoplus_{\chi \in \mathrm{X}(N)} V_\chi$$

By Lemma 1.5.1.4, this is a direct sum. Moreover, we saw in Lemma 1.5.1.5 that, since N is a normal subgroup of G, the subspace V' is stable under $\varphi(G)$, and so we may as well replace V by V', so we can assume $V = \bigoplus V_\chi$.

We define a subspace $W \subset \mathrm{End}V$ as follows:

$$W = \{f \in \mathrm{End}V \mid f(V_\chi) \subset V_\chi, \text{ for all } \chi \in \mathrm{X}(N)\}$$

Since $\mathrm{End}V = \mathscr{L}(GL(V)$, it follows that $GL(V)$ acts on $\mathrm{End}V$ via the Adjoint representation (see Chapter 9 of [6]). That is,

$$\begin{aligned} GL((,V)) \times \mathrm{End}V &\longrightarrow \mathrm{End}V \\ (g,f) &\longrightarrow \mathrm{Ad}g(f) \end{aligned}$$

This action induces an action of $\varphi(G)$ on W:

$$\begin{aligned} \varphi(G) \times W &\longrightarrow W \\ (\varphi(x), f) &\longrightarrow \mathrm{Ad}\varphi(g)(f) \end{aligned}$$

We use this action to define a representation $\psi : G \to GL(W)$ as follows:

$$\begin{aligned} G &\longrightarrow GL(W) \\ x &\longmapsto \mathrm{Ad}\varphi(x)|_W \end{aligned}$$

Since ψ is the composition of φ followed by Ad, and then restricted to W, it is immediate that it is a morphism of algebraic groups. We wish to calculate its kernel.

Suppose $x \in N$. Then $\varphi(x)v = \chi(x)v$ for each $v \in V_\chi$ by the definition of the weight space. That is, $\varphi(x)$ acts on vectors in V_χ by scalar multiplication. But this then implies that, if $f \in W$ and $v \in V_\chi$,

$$\mathrm{Ad}\varphi(x)f(v) = f(v)$$

since $f(v) \in V_\chi$ by the construction of W. Indeed, the above equation holds since $\mathrm{Ad}\varphi(x)$ is ultimately a conjugation operation in this context where $\varphi(G) \subset GL(V)$; see Lemma 10.3B in [6]. We have shown, then, that $\psi(x)f = \mathrm{Ad}\varphi(x)(f) = f$ for all $f \in W$, and therefore $x \in \ker\varphi$.

On the other hand, suppose $x \in \ker\varphi$. Then, for all $f \in W$, we have $\mathrm{Ad}\varphi(x)f = f$, which means that $\mathrm{Ad}\varphi(x)$ commutes with all the elements of $f \in \mathrm{End}V_\chi$. But the only maps which commute with all the elements of $\mathrm{End}V_\chi$ are the scalars, and so $\varphi(x)$ acts as scalar multiplication on V_χ for each $\chi \in \mathrm{X}(N)$. In particular, this holds for the one dimensional subspace $L = \chi_0$, which is to say $\varphi(x)L = L$, and so $x \in N$ by the definition of φ.

We have therefore demonstrated that $N = \ker \varphi$. By Theorem 1.5.3.2, G/N is a quotient of G by N, but we have just shown that $G/N = \psi(G)$, which, by Proposition 1.3.2.1(2) is a closed subvariety of the affine variety $GL(W)$, and so G/N is affine, as required.

The proof of (3) is almost identical to that of (2), and it hinges on the fact that $d\mathrm{Ad} = \mathrm{ad}$, a fact which is given as Theorem 10.4 in [6]. Recall that, for $\mathsf{x}, \mathsf{y} \in \mathfrak{g}$, then $\mathrm{ad}(\mathsf{x})\mathsf{y} = [\mathscr{L}, \mathsf{y}]$. If $\mathsf{x} \in \mathscr{L}(N)$, then $d\varphi(\mathsf{x})v = d\chi(x)v$ for all $v \in V_\chi$, that is, $d\varphi(\mathsf{x})$ acts by scalar multiplication, and so $[d\varphi(\mathsf{x}), \mathsf{y}] = 0$ for all $\mathsf{y} \in \mathfrak{gl}(W)$. Therefore, $\mathsf{x} \in \ker(\mathrm{ad}(d\varphi)) = \ker(d\psi)$. On the other hand, if $\mathsf{x} \in \ker(d\psi)$, then $\mathrm{ad}(d\varphi)(\mathsf{y}) = 0$ for all $\mathsf{y} \in \mathfrak{gl}(W)$, and so $d\varphi(\mathsf{x})$ commutes with all of the elements of $\mathrm{End}(V_\chi)$. But again this implies that $d\varphi(\mathsf{x})L \subset L$, and so $\mathsf{x} \in \mathscr{L}(N)$ by the definition of φ. $\square$

Chapter 2

Solvable Groups

We will see in Chapter 3 that a large part of studying the structure of an algebraic group is concerned with studying a maximal subgroup which is *solvable*. Solvable groups behave nicely, and thus help by focusing on them we can reduce somewhat the complexity of the general case. The aim of this chapter, then, is to discover just how nicely they do behave. In particular, the destination for this chapter is a structure theory for solvable groups, summarised as Theorem 2.3.4.4, which says they can always be decomposed into a unipotent part and a toric part. Broadly speaking, the steps along the way, then, serve to understand a little about these toric and unipotent parts.

We should also note here that another key theorem of this chapter is the Lie Kolchin theorem, given as Theorem 2.2.5.1, which says, roughly, that all solvable groups look like a collection of upper triangular matrices. This theorem is particularly useful when it comes to analysing the Classical Groups in Chapter 5.

2.1 d-Groups

The first stage of our analysis of solvable groups is to construct a notion of a *torus*, and to discern some of its properties. Ultimately, the maximal tori of a reductive group play a crucial role in the structure theory.

2.1.1 Commutative Groups

We first introduce some notation. $D(n,k)$ denotes the group of all invertible diagonal matrices, and its Lie algebra $\mathfrak{d}(n,K)$ consists of *all* diagonal matrices. Similarly, $T(n,K)$ denotes the group of all invertible upper triangular matrices (that is, with 0's below the diagonal), and its Lie algebra $\mathfrak{t}(n,K)$ consists of *all* upper triangular matrices.

Definition 2.1.1.1. A subset of matrices $M \subset \mathrm{Mat}(n,k)$ is *diagonalisable* if there exists a matrix $x \in GL(n, K)$ such that

$$xMx^{-1} \subset \mathfrak{d}(n, K)$$

M is *trigonalisable* if there exists a matrix $x \in GL(n, K)$ such that

$$xMx^{-1} \subset \mathfrak{t}(n, K)$$

Proposition 2.1.1.2. *If $M \subset \mathrm{Mat}(n, K)$ is a set consisting of matrices which commute with each other, then M is trigonalisable. If M consists also of semisimple elements, then M is diagonalisable.*

Proof. We will prove the first result by induction on n, so assume M is commutative. In the case of $n = 1$, the assertion is trivial, so we may assume it holds for all dimensions less than n. If M consists of scalar matrices, then we are done. Suppose, on the other hand, there exists a nonscalar $g \in M$ with eigenvalue $\lambda \in K$. Then $W = \ker(g - \lambda \cdot 1)$ is a nonzero proper subspace of K^n. Let $h \in M, w \in W$. Then

$$(g - \lambda.1)(hw) = ghw - \lambda hw = h(gw - \lambda w) = 0$$

so $hw \in W$, which is to say W is M-stable. Hence, we may consider $M \subset \mathrm{Mat}(m, K)$ where $\dim W = m < n$. Applying the induction hypothesis tells us that M is trigonalisable in $\mathrm{Mat}(m, K)$, which is to say, there exists a basis $\{v_1, \ldots, v_m\}$ of W such that the subspace $\mathrm{span}\{v_1, \ldots, v_i\}$ is M-stable for each $1 \leq i \leq m$. In particular, Kv_1 is M-stable. Consider now the quotient space $U = V/Kv_1$. Since Kv_1 is M-stable, it follows immediately that U is M-stable. Moreover, $\dim U = \dim V - 1$, and so the induction hypothesis applies to U, which is to say there exists a basis $\{u_2 + Kv_1, \ldots, u_n + Kv_1\}$ of U such that the subspace $\mathrm{span}\{u_2 + Kv_1, \ldots, u_i + Kv_1\}$ is M-stable for each $2 \leq i \leq n$. It follows that $\{v_1, u_2, \ldots, u_n\}$ is a basis for V such that the span of the first i elements is M-stable, which is to say M is trigonalisable.

We now prove the second part of the statement, so assume M is commutative, and consists of semisimple elements. Once more we use induction on n, and the case $n = 1$ is trivial, so we can assume that the result holds for dimension less than n. Again, if M consists of scalar matrices, we are done, so suppose otherwise. We can therefore choose an nonscalar element $g \in M$, with eigenvalue λ, such that $W = \ker(g - \lambda \cdot 1)$ is a nonzero, proper subspace of K^n. By the same argument as above, W is M-stable, with $\dim W = m < n$. Since g is semisimple, its eigenspaces span K^n. That is,

$$K^n = W \oplus W' \tag{2.1}$$

where W' is a nonzero sum of eigenspaces of g. By the induction hypothesis, then, there exists a basis $\{v_1, \ldots, v_m\}$ of W such that the subspace Kv_i is M-stable. Moreover, since W' is a nonzero proper subspace of K^n, we can similarly find a basis $\{u_{m+1}, \ldots, u_n\}$ of W' such that Ku_j is M-stable. Equation (2.1) shows that $\{v_1, \ldots v_m, u_{m+1}, \ldots, u_n\}$ is a basis of K^n such that each basis vector spans a one-dimensional M-stable subspace, which shows that M is diagonalisable, as required. $\square$

Definition 2.1.1.3. Let G be an algebraic group. Denote by G_u the subset of G consisting of its unipotent elements. Similarly, denote by G_s the subset of G consisting of its semisimple elements.

Theorem 2.1.1.4. *Let G be a commutative algebraic group. Then G_s and G_u are closed subgroups, are connected if G is, and the product map*

$$\phi : G_s \times G_u \longrightarrow G$$

is an isomorphism of algebraic groups.

Proof. Certainly $e \in G_s \cap G_u$, so these sets are nonempty. Let $x, y \in G_s$. Then it is a consequence of the Jordan decomposition (see, for example, Lemma 15.1B of [6]) that, since $xy = yx$, we have $(xy)_s = x_s y_s = xy$, which implies $xy \in G_s$. Therefore G_s is closed under multiplication, and a similar argument applies to G_u. Suppose $x \in G_s$ such that $xy = e$. Then $e = (xy)_s = xy_s$, which implies $y_s = x^{-1} = y$, and so $y \in G_s$, which shows G_s is closed under taking of inverses, and again a similar argument holds for G_u. Therefore G_s and G_u are subgroups.

Let U be the set of all unipotent matrices in $GL(n, K)$. Then $U = \{x \in GL(n, K) \mid (x-1)^m = 0, m \leq n\}$, and so is closed. But there exists a morphism $\rho : G \to GL(n, K)$, and $G_u = \rho^{-1}(U)$, which implies G_u is closed, too.

Let $x_1, x_2 \in G_s, y_1, y_2 \in G_u$. Then

$$\begin{aligned}\varphi((x_1, y_1)(x_2, y_2)) &= \varphi(x_1x_2, y_1y_2)\\ &= x_1x_2y_1y_2 = x_1y_1x_2y_2 = \varphi(x_1, y_1)\varphi(x_2, y_2)\end{aligned}$$

where $x_2y_1 = y_1x_2$ since the group G is commutative. This shows that φ is a group homomorphism. Moreover, if $\varphi(x, y) = xy = e$, then $x, y \in G_s \cap G_u = \{e\}$, which shows that φ is injective, and the fact that any element $g \in G$ has a decomposition $g = g_s g_u = \varphi(g_s, g_u)$ shows that φ is surjective, and therefore that φ is a group isomorphism.

Embedding G in $GL(n, K)$ and applying Proposition 2.1.1.2 shows us that, without loss of generality, we can assume $G \subset T(n, K)$, and $G_s \subset D(n, K)$. Certainly $G \cap D(n, K)$ consists of diagonlisable elements, and so $G \cap D(n, K) \subset G_s$. On the other hand, $G_s \subset D(n, K) \subset G \cap D(n, K)$, so $G_s = G \cap D(n, K)$, and so is closed. It follows that G_s, G_u are algebraic groups, and so the product map φ is a morphism of algebraic groups. It remains to show that φ^{-1} is a morphism of algebraic groups.

We consider two maps, $f : G \to G_s$ which sends $x \mapsto x_s$ and $g : G \to G_u$ which sends $x \mapsto x_u$. Note that $\varphi^{-1} = f \times g$, so if we show that f and g are morphisms, then it will follow that φ^{-1} is, too.

Let $x \in G$, and let x_d denote the diagonal part of x. Since $x \in T(n, K)$, by assumption, it is easy to see that x and x_d have the same eigenvalues and mutliplicities. But since $f(x) = x_s$ has the same eigenvalues and multiplicities again and is also diagonal, it is, at worst, a rearrangement of the entries of x_d. The map which sends x to x_d is clearly a morphism, and so this map, followed by the appropriate rearrangment, is equal to f and in particular, f is a morphism. Note now that $g(x) = f(x)^{-1}x$, and so g must be a morphism, too. An immediate consequence of this is that G_s, G_u are connected if G is, being the images under the morphisms f, g respectively. □

2.1.2 Diagonalisable Groups

Definition 2.1.2.1. An algebraic group G is *diagonalisable* if it is isomorphic to a closed subgroup of $D(n, K)$ for some n.

Lemma 2.1.2.2. *If G is diagonalisable, then G is commutative and consists of semisimple elements.*

Proof. Let $\varphi : G \to H$ be an isomorphism, where H is a closed subgroup of $D(n, K)$. Since $D(n, K)$ is commutative, so is H, and therefore G. Similarly, since $D(n, K)$ consists of semisimple elements, so does H, and since morphisms map semisimple elements to semisimple elements, we have that G consists of semisimple elements, too. □

Proposition 2.1.2.3. *If G is commutative and consists of semisimple elements, then G is conjugate in $GL(n, K)$ to a subgroup of $D(n, K)$.*

Proof. This is a corollary of Proposition 2.1.1.2. Let $\varphi : G \to H$ be an isomorphism, with H a subgroup of $GL(n, K)$. If G is commutative and consists of semisimple elements, then the same is true of H, and so Proposition 2.1.1.2 tells us that there is an element $x \in GL(n, K)$ such that xHx^{-1} consists of diagonal matrices. But since the elements of H are invertible, so are the elements of xHx^{-1}, and so xHx^{-1} is a subset, and indeed a subgroup, of $D(n, K)$, as required. □

Corollary 2.1.2.4. *A group G is diagonalisable if and only if it is commutative and consists of semisimple elements.*

Proof. If a group G is conjugate in $GL(n, K)$ to a subgroup H of $D(n, K)$, then G is certainly isomorphic to H. Thus we can immediately combine Lemma 2.1.2.2 and Proposition 2.1.2.3 to get the result. □

Proposition 2.1.2.5. *Closed subgroups and homomorphic images of a diagonalisable group are again diagonalisable.*

Proof. Let H be a closed subgroup of G, and let $\psi : G \hookrightarrow D(n, K)$ be an embedding of G into $D(n, K)$. Then $\psi|_H$ is clearly also an embedding, and so H is also diagonalisable. Suppose $\varphi : G \to G'$ is a homomorphism. Then, by Lemma 2.1.2.2, G is commutative and consists of semisimple elements, so the same is true of $\varphi(G)$. Therefore, according to Proposition 2.1.2.3, $\varphi(G)$ is conjugate (and therefore isomorphic) to a subgroup of $D(n, K)$, and so is diagonalisable. □

2.1.3 Characters

Proposition 2.1.3.1. *Let G be an abstract group, X the set of homomorphisms $G \to K^*$. Then X is a linearly independent subset of the space of all K-valued functions on G.*

Proof. We argue by contradiction, so suppose that $n > 1$ is minimal such that $\chi_1, \ldots \chi_n \in \mathrm{X}$ are distinct and linearly dependent, that is,

$$\sum_{i=1}^{n-1} a_i \chi_i + \chi_n = 0$$

for $a_i \in K$. Not all the a_i are zero, so we may suppose, without loss of generality, that $a_1 \neq 0$. Since $\chi_1 \neq \chi_n$, choose $y \in G$ such that $\chi_1(y) \neq \chi_n(y)$. Now, for arbitrary $x \in G$, consider the equations

$$0 = \sum_{i=1}^{n-1} a_i \chi_i(xy) + \chi_n(xy) = \sum_{i=1}^{n-1} a_i \chi_i(x)\chi_i(y) + \chi_n(x)\chi_n(y) \tag{2.2}$$

and

$$0 = \left(\sum_{i=1}^{n-1} a_i \chi_i(x) + \chi_n(x) \right) \chi_n(y) = \sum_{i=1}^{n-1} a_i \chi_i(x)\chi_n(y) + \chi_n(x)\chi_n(y) \tag{2.3}$$

If we subtract equation (2.3) from (2.2) we get

$$0 = \sum_{i=1}^{n-1} \big(a_i(\chi_i(y) - \chi_n(y))\big)\chi_i(x)$$

with not all coefficients equal to 0, that is, at least $a_1(\chi_1(y) - \chi_n(y)) \neq 0$. Note firstly that, if $n = 2$, then this implies

$$a_1(\chi_1(y) - \chi_2(y))\chi_1(x) = 0$$

But $a_1 \neq 0$ and $\chi_1(y) \neq \chi_2(y)$, which forces $\chi_1(x) = 0$, which is absurd. Suppose then $n > 2$. In this case, since x was arbitrary, we have that the elements $\chi_1, \ldots, \chi_{n-1}$ are distinct and linearly dependent, which contradicts the minimality of n. $\square$

Definition 2.1.3.2. A *(rational) character* of an algebraic group G is a morphism of algebraic groups

$$\chi : G \to \mathbb{G}_m$$

We denote by $\mathrm{X}(G)$ the set of all characters of G.

Note that we can consider $\mathrm{X}(G) \subset K[G]$, since $\mathbb{G}_m \subset K$.

Proposition 2.1.3.3. *The set $\mathrm{X}(G)$ is a linearly independent subset of the space of all K-valued functions on G.*

Proof. Since $X(G)$ consists of homomorphisms $G \to K^*$, it is a subset of the set X of all such homomorphisms. By Proposition 2.1.3.1, then, $X(G)$ is a linearly independent set. $\square$

Proposition 2.1.3.4. *The set* $X(G)$ *is an abelian group under pointwise multiplication.*

Proof. Suppose $\chi, \psi \in X(G)$, then $(\chi\psi)(x) = \chi(x)\psi(x) \in K^*$ defines another character, and so $X(G)$ is closed under multiplication, a multiplication which is clearly commutative. Secondly, given $\chi \in X(G)$, we can define an inverse $\psi(x) = 1/\chi(x)$, which is always defined since the codomain of χ is K^*. So $\psi \in X(G)$ and $X(G)$ is closed under the taking of inverses. $\square$

Proposition 2.1.3.5. *The character group* $X\big(D(n,K)\big)$ *is free abelian of rank* n *with basis consisting of the coordinate functions* $\chi_1, \dots \chi_n$*, where*

$$\chi_i : \operatorname{diag}(a_1, \dots a_n) \longmapsto a_i$$

Proof. Note firstly that every monic monomial over K in n indeterminates is a character of D. Indeed, let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^l$. Then we can define a character $\chi_{\mathbf{a}} \in X(D)$ as follows:

$$\begin{aligned} \chi_{\mathbf{a}} : D &\longrightarrow \mathbb{G}_m \\ \operatorname{diag}(x_1, \dots, x_n) &\longmapsto x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} \end{aligned}$$

Certainly $\chi_{\mathbf{a}}$ is a character, since it is a homomorphism and $x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$ is polynomial in the matrix entries of elements of D.

By definition, elements of $X(D)$ are morphisms of varieties, and therefore are polynomial functions in the n (nonzero) matrix entries of elements of D. But every such polynomial function in is a linear combination of monic monomial functions, and therefore since $X(D)$ is a linearly independent subset of the space of all K-valued functions on D by Proposition 2.1.3.3, it consists only of monomial functions

We now define a map

$$\begin{aligned} f : \mathbb{Z}^n &\longrightarrow X(D) \\ \mathbf{a} &\longmapsto \chi_{\mathbf{a}} \end{aligned}$$

We aim to show that f is an isomorphism of groups. Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, and $x = \operatorname{diag}(x_1, \dots, x_n)$. Then

$$\begin{aligned} \chi_{\mathbf{a}+\mathbf{b}}(x) &= x_1^{a_1+b_1} \cdots x_n^{a_n+b_n} \\ &= x_1^{a_1} x_1^{b_1} \cdots x_n^{a_n} x_n^{b_n} \\ &= x_1^{a_1} \cdots x_n^{a_n} x_1^{b_1} \cdots x_n^{b_n} \\ &= \chi_{\mathbf{a}}(x)\chi_{\mathbf{b}}(x) \\ &= \chi_{\mathbf{a}} \cdot \chi_{\mathbf{b}}(x) \end{aligned}$$

So $f(\mathbf{a}+\mathbf{b}) = \chi_{\mathbf{a}} \cdot \chi_{\mathbf{b}}(x) = f(\mathbf{a})f(\mathbf{b})$. So f is a group homomorphism.

Suppose now that $f(\mathbf{a}) = 1$. That is, suppose $x_1^{a_1} \cdots x_n^{a_n} = 1$ for all $x_i \in K^*$. It follows immediately that $a_1 = \cdots = a_n = 0$, so f is injective. Suppose now that $\chi \in \mathrm{X}(D)$, and let $x = \mathrm{diag}(x_1, \ldots, x_n) \in D$. Certainly, the following map

$$\begin{aligned} \chi_i : D &\longrightarrow \mathbb{G}_m \\ x &\longmapsto x_i \end{aligned}$$

is a character.

We showed above that every element of $\mathrm{X}(D)$ is a monic monomial, and therefore of the form $\chi_{\mathbf{a}} = f(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{Z}$. This tells us that f is surjective, and so $\mathrm{X}(D) \cong \mathbb{Z}^n$, as required. □

2.1.4 d-Groups

The following terminology is taken from Chapter 16 of [6]:

Definition 2.1.4.1. An algebraic group G is a *d-group* if $K[G]$ has a basis (as K-algebra) consisting of characters of G.

Proposition 2.1.4.2. *An algebraic group G is a d-group if and only if $\mathrm{X}(G)$ is a basis of $K[G]$.*

Proof. If G is a d-group, then $K[G]$ has a basis consisting of characters. Suppose that $\chi \in \mathrm{X}(G)$, but is not an element of this basis. Since $\chi \in K[G]$, this implies that χ is a linear combination of other characters, which contradicts Proposition 2.1.3.1, so χ must be included in this basis, and so $\mathrm{X}(G)$ must be a basis for $K[G]$. The reverse direction is by definition. □

Example 14. We saw in the proof of Proposition 2.1.3.5 that $D(n, K)$ is a d-group.

Proposition 2.1.4.3. *1. Let G be a d-group. If H is a closed subgroup of G, then H is also a d-group, and is the intersection of kernels of some characters of G. That is, there is some subset $\mathrm{Z} \subset \mathrm{X}(G)$*

$$H = \bigcap_{\chi \in \mathrm{Z}} \ker \chi$$

2. In particular, diagonalisable groups are d-groups.

3. Any d-group is diagonalisable

Proof. For (1), let $\varphi : H \hookrightarrow G$ be the inclusion morphism. Since $\varphi^*(f) = f \circ \varphi$ it is easy to see that $\varphi^*(f) = f|_H$. Now φ^* is clearly surjective, since, for $h \in K[H]$, we have $h = h|_H = \varphi^*(h)$. Similarly, for $\chi \in \mathrm{X}(G)$, we have $\chi|_H \in \mathrm{X}(H)$. Now, suppose $h \in K[H]$. Then $h \in K[G]$ and so, since G is a d-group, $h = \lambda_1\chi_1 + \cdots + \lambda_m\chi_m$ for some $\chi_i \in \mathrm{X}(G)$. Then $h = \lambda_1\chi_1|_H + \cdots + \lambda_m\chi_m|_H$, so $\mathrm{X}(H)$ spans $K[H]$. Moreover, by another application of Proposition 2.1.3.1, we know that $\mathrm{X}(H)$ is linearly independent, so $\mathrm{X}(H)$ is a basis for $K[H]$, that is, H is a d-group.

Suppose now that $f \in \mathscr{I}(H)$. Since $f \in K[G]$, we have $f = \sum a_i\chi_i$ for $\chi_i \in \mathrm{X}(G)$. If we restrict both sides of this equation to H, then we get

$$0 = \sum a_i\chi_i|_H$$

Applying Proposition 2.1.3.1 to the group H tells us that those characters in the above equation which are equal on H have coefficients summing to 0. That is, $\mathscr{I}(H)$ is spanned by elements $g = \sum b_j\chi_j$, where all the χ_j's are equal on H, and where $\sum b_j = 0$.

Consider the following expression:

$$\chi_1\big(\sum b_j(\chi_1^{-1}\chi_j - 1)\big) = (\sum b_j\chi_j) - \chi_1\sum b_j = \sum b_j\chi_j$$

where the last equality is since $\sum b_j = 0$. So we can rewrite the spanning elements g above as $g = \chi_i g'$, where $g' = \sum b_i(\chi_1^{-1}\chi_j - 1)$. Since g' is a linear sum of functions $\psi - 1$, where $\psi = \chi_1^{-1}\chi_j \in \mathrm{X}(G)$, we have shown that $\mathscr{I}(H)$ is generated as an ideal by expressions of the form $\psi - 1$. Therefore H consists of those elements which vanish on all of these expressions, and so H consists of elements which lie in the intersection of all of the $\ker\varphi$, as required.

Now (2) follows from (1), given that Example 14 told us that $D(n,K)$ is a d-group.

Finally, for (3), since $K[G]$ is a finitely generated K-algebra (since G is an affine variety), and since $\mathrm{X}(G)$ spans $K[G]$, we can find finitely many characters, $\chi_1,\dots,\chi_n$, say, which generate $K[G]$ as K-algebra. We then define a map

$$\begin{aligned}\varphi : G &\longrightarrow \mathbb{G}_m \times \cdots \times \mathbb{G}_m\\ g &\longmapsto (\chi_1(g),\dots,\chi_n(g))\end{aligned}$$

Now φ is certainly a morphism of varieties, since the χ_i are. We wish to show that $\ker\varphi$ is trivial. Indeed, suppose $1 \neq g \in \ker\varphi$. Then we choose an element $f \in K[G]$ such that $f(g) \neq f(1)$. Such an element certainly exists, and, moreover, there exist elements $a_i \in K$ such that $f = \sum a_i\chi_i$, by our choice of basis. But then

$$f(g) = \sum a_i\chi_i(g) = \sum a_i = f(1)$$

which is a contradiction. Therefore $\ker\varphi$ is trivial, and so $G \cong \mathrm{im}\varphi \subset \mathbb{G}_m \times \cdots \times \mathbb{G}_m$, and since $\mathbb{G}_m \times \cdots \times \mathbb{G}_m \cong D(n,K)$, we know that G consists of commutative and semisimple elements. Applying Proposition 2.1.2.3, we get that G is diagonalisable. $\square$

In particular, the above proposition tells us that

Corollary 2.1.4.4. *An algebraic group G is diagonalisable if and only if it is a d-group.* $\square$

2.1.5 The Structure of a d-Group

The main result of this section requires the following (general) lemma:

Lemma 2.1.5.1 (The Splitting Lemma). *Let A, B, C be objects in an abelian category, such that, for maps $q : A \to B, r : B \to C$, the sequence*

$$A \xrightarrow{q} B \xrightarrow{r} C$$

is short exact. Then the following statements are equivalent:

1. *There exists a map $t : B \to A$ such that $t \circ q = \mathrm{id}_A$.*
2. *There exists a map $u : C \to B$ such that $r \circ u = \mathrm{id}_C$.*
3. *$B \cong A \oplus C$, and, furthermore, q is the natural injection of A into B, and r is the natural projection of B onto C.*

If any of these hold, the sequence is said to be split.

Proof. First we show (1) implies (3). We begin by observing that $B \subset \mathrm{im}q \oplus \ker t$. Indeed, if $b \in B$, then certainly $b = (b - qt(b)) + qt(b)$. Clearly the last term of this equation lies in $\mathrm{im}q$, and since $t(b - qt(b)) = t(b) - t(b)$, the first term lies in $\ker t$. Furthermore, $\mathrm{im}q \cap \ker t = \{e\}$. Indeed, if $b \in \mathrm{im}q \cap \ker t$, then $b = q(a)$ for some $a \in A$, and $t(b) = 0$. But $a = tq(a) = t(b) = 0$, and so $b = q(0) = 0$. Now it is easy to show $B = A \oplus C$. Indeed, we define a morphism

$$\begin{aligned} \alpha : B &\longrightarrow A \oplus C \\ b &\longmapsto (t(b), r(b)) \end{aligned}$$

First we show injectivity. Suppose $b \in \ker \alpha$. Then $r(b) = 0$, so, by exactness, there is an element $a \in A$ such that $q(a) = b$. But $a = tq(a) = t(b) = 0$, so $b = q(0) = 0$, as required. For surjectivity, suppose $(a, c) \in A \oplus C$. Now $a = tq(a)$, and since r is surjective, there is an element $b \in B$ such that $r(b) = c$. We now ust the above argument to decompose $b = b_1 + b_2$ such that $b_1 \in \ker t$ and $b_2 \in \ker r = \mathrm{im}q$, then $r(b) = r(b_1) + r(b_2) = r(b_1) = c$. So

$$\alpha(q(a) + b_1) = \big(t(q(a) + b_1), r(q(a) + b_1)\big) = (a + 0, 0 + c)$$

as required.

To show (2) implies (3) uses a similar argument. First we have that $B \subset \ker r \oplus \mathrm{im}u$, since for any $b \in B$, $b = (b - ur(b)) + ur(b)$. Suppose $b \in \ker r \cap \mathrm{im}u$, with $u(c) = b$, say. Then $c = ru(c) = r(b) = 0$, so $b = u(0) = 0$, thus $\ker r \cap \mathrm{im}u = 0$, and so $B \subset \ker r \oplus \mathrm{im}u = \mathrm{im}q \oplus \mathrm{im}u$. Now we define a map

$$\begin{aligned} \beta : A \oplus C &\longrightarrow B \\ (a, c) &\longmapsto q(a) + u(c) \end{aligned}$$

We need to show this is an isomorphism. Surjectivity is immediate from the fact that $B \subset \mathrm{im}q \oplus \mathrm{im}u$. For injectivity, suppose $\beta(a, c) = 0$, that is, $q(a) = -u(c)$.

Then $q(a) \in \text{im}q \cap \text{im}u$, but we saw above that this intersection is trivial, and so $q(a) = u(c) = 0$. But since both q and u are injective (the latter since $ru = \text{id}$), this implies $a = c = 0$, as required.

Finally, that (3) implies both (1) and (2), one need only define t to be the natural projection of $A \oplus C$ onto A, and u to be the natural injection of C into $A \oplus C$. □

Note that Lemma 2.1.5.1 does not hold for non-abelain objects, like non-abelian groups, in this form. We give a non-abelian group formation later, as Lemma 2.3.2.1. The problem for non-abelian groups is that the group $u(C)$ might not be normal, and therefore the direct sum (or direct product, as this is equivalent in the category of abelian groups) is not defined.

Lemma 2.1.5.2. *Let G be a d-group. Then $\mathrm{X}(G)$ is a finitely generated abelian group.*

Proof. Since G is diagonalisable, by Proposition 2.1.4.3, we have an embedding $\varphi : G \hookrightarrow D(n, K)$ for some n. We then consider $\varphi^\circ : \mathrm{X}(D(n, K)) \to \mathrm{X}(G)$, which is the restriction homomorphism, to realise that since $\mathrm{X}(D(n, K)) = \mathbb{Z}^n$, then $\mathrm{X}(G)$ is a homomorphic image of $\mathbb{Z}^n$, and so is finitely generated abelian. □

Recall from group theory that, if A is a finitely generated abelian group, then

$$A = \mathbb{Z}^r \times B$$

where B is the *torsion subgroup* of A, that is, the set of all elements of A with finite order. In addition, r is the *rank* of A, a well-defined non negative integer.

Lemma 2.1.5.3. *If G is a connected algebraic group, then $\mathrm{X}(G)$ is torsion free.*

Proof. Let $\chi \in \mathrm{X}(G)$. Then χ is a morphism of algebraic groups, and so the image $\chi(G)$ is a connected subgroup of $\mathbb{G}_m$. But such a subgroup must be of dimension less than or equal to 1, and so the only such groups are 1 and $\mathbb{G}_m$ itself.

Suppose $\chi^m = 1$ for some m, which is to say $\chi^m(g) = 1$ for all $g \in G$. If any element $g \in G$ is such that $\chi(g) \neq 1$, then we saw above that χ is surjective, and therefore this implies that all the elements of $\mathbb{G}_m$ are of finite order. But this is clearly absurd, and so $\chi(g) = 1$ for all $g \in G$, that is $\chi = 1$. Therefore, $\mathrm{X}(G)$ is torsion free. □

Example 15. We saw in Proposition 2.1.3.5 that $\mathrm{X}\big(D(n, K)\big) \cong \mathbb{Z}^n$. It can also be shown that $\mathrm{X}\big(GL(n, K)\big) = \mathbb{Z}$, and $\mathrm{X}\big(SL(n, K)\big) = \{e\}$

Definition 2.1.5.4. An algebraic group G is called a *torus* if it is isomorphic to $D(n, K)$ for some n.

This of course means that all tori are d-groups.

Theorem 2.1.5.5. *Let G be a d-group. Then*

$$G = G^\circ \times H \qquad \textit{a direct product of algebraic groups}$$

where G° is a torus, and H is a (not necessarily uniquely determined) finite group. In particular, a connected d-group is a torus.

Proof. Since G is a d-group, we can consider it to be a subgroup of $D = D(n, K)$, by Proposition 2.1.4.3. Moreover, we know that $\mathrm{X}(G^\circ)$ is torsion free by Lemma 2.1.5.3, and finitely generated abelian by Lemma 2.1.5.2, so $\mathrm{X}(G^\circ) = \mathbb{Z}^r$, while $\mathrm{X}(D) = \mathbb{Z}^n$ by Proposition 2.1.3.5. Moreover, we have a surjective restriction group homomorphism

$$\mathbb{Z}^n = \mathrm{X}(D) \xrightarrow{\varphi^\circ} \mathrm{X}(G^\circ) = \mathbb{Z}^r$$

We consider the short exact sequence

$$0 \longrightarrow \ker\varphi^\circ \longrightarrow \mathrm{X}(D) \xrightarrow{\varphi^\circ} \mathrm{X}(G^\circ) \longrightarrow 0 \tag{2.4}$$

and we wish to apply Lemma 2.1.5.1 to it.

Recall that a free $\mathbb{Z}$-module is a projective module. A defining property of projective modules is a lifting property, which states that, given projective modules M, N, P, a surjective morphism $f : N \to M$, and an arbitrary morphism $g : P \to M$, there exists a morphism $h : P \to N$, as in the following diagram:

$$\begin{tikzcd} & N \arrow[d, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

Note that h need not be unique, and hence this is not a *universal* property. Nevertheless, We are in precisely this situation, if we let $M, P = \mathrm{X}(G^\circ)$ and $N = \mathrm{X}(D)$, and set $f = \varphi^\circ$, and $g = \mathrm{id}$. This guarantees the existence of a map $h : \mathrm{X}(G^\circ) \to \mathrm{X}(D)$ such that $f \circ h = \mathrm{id}$, and so we can apply the splitting lemma to the sequence (2.4) to get $\mathrm{X}(D) = \ker\varphi^\circ \oplus \mathrm{X}(G^\circ)$. Now, since $\ker\varphi^\circ$ is a subgroup of the free abelian group $\mathrm{X}(D)$ it has a basis $\chi_1, \ldots, \chi_{n-r}$ to which we can add a basis $\chi_{n-r+1}, \ldots, \chi_n$ of $\mathrm{X}(G^\circ)$. Now consider the homomorphism

$$\begin{aligned} \alpha : D &\longrightarrow D \\ x &\longmapsto \mathrm{diag}(\chi_1(x), \ldots, \chi_n(x)) \end{aligned}$$

Since the $\chi_1, \ldots, \chi_n$ form a basis, α is clearly injective. Furthermore, we know from the proof of Lemma 2.1.5.3 that nontrivial characters of a connected group are surjective, so α is surjective, and therefore is an automorphism of D. Moreover, α sends G° to the subgroup of D consisting of diagonal matrices with 1's in the first $n - r$ coordinates. These facts combined show that G° is a torus. Now, if we define the subgroup

$$D' = \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_{n-r+1} = \cdots = a_n = 1\}$$

then clearly $D = G^\circ \times D'$, as algebraic groups. Now we define a group $H = G \cap D' \cong G/G^\circ$, where the isomorphism is, for example, by the map

$$\begin{aligned} \gamma : G/G^\circ &\longrightarrow G \cap D' \\ xG^\circ &\longmapsto x \end{aligned}$$

Putting these facts together, we have $G = G^\circ \times H$. Since G° has finite index in G, it is clear that H is finite. $\square$

We also offer the following crucial result:

Proposition 2.1.5.6. *Any rational representation of a diagonalisable group G is a direct sum of finitely many 1-dimensional ones. In particular, if $\rho : G \to GL(V)$, then*

$$V = \bigoplus_{\chi \in \mathrm{X}(T)} V_\chi$$

where the spaces

$$V_\chi = \{v \in V \mid \rho(g)v = \chi(g)v, \forall g \in G\}$$

are linearly independent and only finitely many of them are nonzero. On the other hand, if G is a group such that every rational representation of G is a direct sum of 1-dimensional representations, then G is diagonalisable.

Proof. We showed in Lemma 1.5.1.4 that the V_χ's are linearly independent, so we move straight on to the decomposition. Let $\varphi : G \to GL(V)$ be a rational representation. Now, for $g \in G$, $\varphi(g) \in GL(V)$, and so, if we construe this in matrix terms, each entry of the matrix $\varphi(g)$ can be written as a linear combination of characters by virtue of the fact that G is diagonalisable, and in particular the fact that the characters form a basis of $K[G]$. Combining the linear combinations of each entry of $\varphi(g)$, we can derive an expression

$$\varphi(g) = \sum_{\chi \in \mathrm{X}(G)} \chi(g) a_\chi \tag{2.5}$$

where $a_\chi \in \mathrm{Mat}(n, K)$ for $n = \dim V$. Now for $g, h \in G$ we have

$$\varphi(gh) = \sum_\chi \chi(gh) a_\chi = \sum_\chi \chi(g)\chi(h) a_\chi \tag{2.6}$$

while on the other hand

$$\varphi(g)\varphi(h) = \left(\sum_\chi \chi(g) a_\chi\right)\left(\sum_{\chi'} \chi'(h) a_{\chi'}\right) \tag{2.7}$$

Since φ is a homomorphism, equations (2.6) and (2.7) are equal. That is,

$$\sum_\chi \chi(g)\left(\chi(h) a_\chi - \sum_{\chi'} \chi'(h) a_\chi a_{\chi'}\right) = 0$$

By Proposition 2.1.3.1, this implies

$$\chi(h)a_\chi - \sum_{\chi'} \chi'(h)a_\chi a_{\chi'} = 0$$

but we can rewrite this as

$$\sum_{\chi'} \chi'(h)\big(\delta_{\chi,\chi'}a_\chi - a_\chi a_{\chi'}\big) = 0$$

where $\delta_{\chi,\chi'}$ is the Kronecker delta, and so another application of Proposition 2.1.3.1 gives us

$$\delta_{\chi,\chi'}a_\chi = a_\chi a_{\chi'} \tag{2.8}$$

We also need the following equality, which we obtain by substituting $g = e$ into equation (2.5):

$$1 = \varphi(e) = \sum_{\chi} \chi(e)a_\chi = \sum_{\chi} a_\chi \tag{2.9}$$

Now define $V_\chi = a_\chi V$. Then equation (2.9) tells us

$$V = \bigoplus_{\chi} V_\chi$$

Furthermore, if we take $g \in G$ and $v \in V_\chi$, that is, $v = a_\chi w$ for some $w \in V$, we get

$$\varphi(g)v = \varphi(g)a_\chi w = \sum_{\chi'} \chi'(g)a_{\chi'}a_\chi w = \chi(g)a_\chi w = \chi(g)v$$

Therefore V is the direct sum of 1-dimensional representations.

Finally we prove the last statement of the proposition. If G is an algebraic group, we can embed G in $GL(n, K)$ for some n by Theorem 1.4.4.8. Taking $V = K^n$ we decompose this representation into 1-dimensional ones, and can therefore represent elements of G as $n \times n$-diagonal matrices. □

2.1.6 Rigidity

The result of this section is that a d-group is 'rigid'. This means that it has 'few' nontrivial automorphisms, and so there are not many ways to permute the group. In particular, the normaliser of a d-group in an ambient algebraic group is not much bigger than the centraliser. That is, most elements of the normaliser yield a trivial automorphism of the group, and hence also lie in the centraliser.

Proposition 2.1.6.1. *Let G be a d-group. Then*

1. *The elements of finite order in G form a dense subset, and*

2. *There are only finitely many elements of G of order m for each $m > 0$.*

Proof. We firstly consider the case $G = \mathbb{G}_m$. If $x \in \mathbb{G}_m = K^*$ is of finite order, then it is necessarily a root of unity. The only elements, therefore, of order m are the m^{th} roots of unity, and there are at most m of these, thus proving (2). Altogether, though, there are certainly infinitely many elements of finite order, since K is algebraically closed. These elements form the torsion subgroup B of $\mathbb{G}_m$, and we wish to consider its closure $\overline{B}$.

Suppose $\overline{B} \subsetneq \mathbb{G}_m$ Then $\dim \overline{B} < \dim \mathbb{G}_m = 1$ and so $\overline{B}$ is finite. But this is absurd, since K is algebraically closed. Therefore $\overline{B} = \mathbb{G}_m$, as required.

$\overline{B} = \mathbb{G}_m$ as required.

For the general case, consider Theorem 2.1.5.5, which shows that $G = \mathbb{G}_m \times \cdots \times \mathbb{G}_m \times H$, where H is finite. Certainly the elements of finite order in G are contained in the set $B \times \cdots \times B \times H$. Now, H is itself finite, and so this, combined with the above argument, shows that $B \times \cdots \times B \times H$ is dense in G, proving (1). Now, for a positive integer m, an element $g \in G$ of order m is of the form $g = (g_1, \cdots, g_k, h)$, where each $g_i \in \mathbb{G}_m$ and $h \in H$ are all themselves of order m. The above argument shows that there can be only finitely many such elements g, which proves (2). □

Proposition 2.1.6.2. *Let $\varphi : X \times G \to H$ be a morphism of varieties such that*

1. *G is an algebraic group whose elements of finite order (that is, its torsion subgroup) form a dense subset.*

2. *H is an algebraic group containing only finitely many elements of order m for each $m > 0$.*

3. *X is a connected variety.*

4. *For each $x \in X$, the map*

$$\begin{aligned} \varphi_x : G &\longrightarrow H \\ y &\longmapsto \varphi(x, y) \end{aligned}$$

   is a group homomorphism.

Then the map $x \to \varphi_x$ is constant. That is, for all $x, x' \in X$, we have $\varphi_x = \varphi_{x'}$, which in turn means that, given any element $y \in G$, we have $\varphi(x, y) = \varphi(x', y)$.

Proof. For each $y \in G$, define a map

$$\begin{aligned} \psi_y : X &\longrightarrow H \\ x &\longmapsto \varphi(x, y) \end{aligned}$$

Then ψ is clearly a morphism of varieties. Moreover, if y has finite order, m, say, then by (4)

$$1 = \varphi_x(y^m) = (\varphi_x(y))^m = (\varphi(x, y))^m = (\psi_y(x))^m$$

so the elements in the image of ψ_y have finite order, less than or equal to m, and so there are only finitely many of them, by (2). But by (3), X is connected, and so $\mathrm{im}\psi_y$ is, too. But if a set is both connected and finite then it must consist of a single element. Therefore, for y of finite order in G, we have $\varphi(x,y) = \varphi(x',y)$, for any $x, x' \in X$; that is, $\varphi_x(y) = \varphi_{x'}(y)$. Again applying (4), we have $\varphi_{x'}\varphi_x^{-1}(y) = e$ for all y in a dense subset of G, dense by (1). But if a continuous map sends a dense subset to a single point, then it sends the whole domain to that point, too. Therefore $\varphi_{x'}\varphi_x^{-1}(y) = e$ for all $y \in G$, and so $\varphi_x = \varphi_{x'}$ as required. □

Corollary 2.1.6.3. *Let D be a diagonalisable subgroup of an algebraic group G. Then*

$$[N_G(D)]^\circ = [C_G(D)]^\circ$$

Proof. Certainly $[C_G(D)]^\circ \subset [N_G(D)]^\circ$, so we just need to show the reverse inclusion. In Proposition 2.1.6.2, set $X = [N_G(D)]^\circ$ and $G = H$ and

$$\begin{aligned} \varphi : X \times G &\longrightarrow G \\ (x,y) &\longmapsto xyx^{-1} \end{aligned}$$

Certainly Proposition 2.1.6.2 (4) holds, as does Proposition 2.1.6.2 (3). Moreover, Proposition 2.1.6.1 (1) and 2.1.6.1 (2) give us the conditions Proposition 2.1.6.2 (1) and 2.1.6.2 (2), so we can apply Proposition 2.1.6.2. In particular, for arbitrary $x \in [N_G(D)]^\circ$, we have $\varphi_x = \varphi_e$, that is $xyx^{-1} = y$ for all y. This says precisely that $[N_G(D)]^\circ \subset [C_G(D)]^\circ$ as required. □

Example 16. We will show in Chapter 5 that

$$N_{GL(n,K)}\big(D(n,K)\big) = \{\text{monomial matrices}\}$$

Recall that a *monomial matrix* is an invertible matrix which has precisely one nonzero entry in each row and each column, that is a diagonal matrix with rows permuted. Note also that, as this example already says, the set of monomial matrices is a subgroup of $GL(n,K)$. The connected component of this normaliser is in fact $D(n,K)$, and so $D(n,K)$ is its own centraliser.

2.2 Solvable and Nilpotent Groups

In this section we introduce the notion of a solvable group, and classify them to an extent, using the Lie Kolchin theorem, a key tool in understanding the greater structure theory of algebraic groups.

2.2.1 Commutator Subgroups

Recall that $(x,y) = xyx^{-1}y^{-1}$ describes the commutator of x and y for $x, y \in G$, and, for subgroups A, B of G, the commutator subgroup (A,B) is the subgroup generated by commutators (x,y), where $x \in A, y \in B$.

Lemma 2.2.1.1. *Let G be an abstract group, with $[G : Z(G)] = n < \infty$, Then (G,G) is a finite group.*

Proof. Write $Z = Z(G)$. The proof is in three stages. The first stage is to show that S, the set of commutators of G, is finite, and in particular has cardinality at most n^2. Secondly, we show that in a product of commutators, any two can be made adjacent. Thirdly we show that the $(n+1)^{th}$ power of any commutator can be rewritten as a product of just n commutators. Combining these three things, we can already see that each element of (G,G) would have at most n^3 factors from S, which certainly implies finiteness of the group.

For stage one, we note that for arbitrary $y \in G$, if $x, x' \in G$ are such that $x' \in xZ$, then $(x,y) = (x',y)$. This is clear, since $x' = xh$, for some central element h, and so $(x',y) = (xh)y(xh)^{-1}y^{-1} = xyx^{-1}y^{-1}$. Similarly, for arbitrary $x \in G$, if $y' \in yZ$, then $(x,y) = (x,y')$.

We can now construct a well-defined map

$$\begin{aligned} G/Z \times G/Z &\longrightarrow S \\ (xZ, yZ) &\longmapsto (x,y) \end{aligned}$$

Since this map is surjective, we have $\mathrm{Card}\big(G/Z \times G/Z\big) \geq \mathrm{Card}(S)$, and therefore $\mathrm{Card}(S) \leq n^2$.

We turn to the second stage, where we show that a product of commutators can always be rewritten so that any two are made adjacent, whilst still being a product of commutators. It suffices to note that, given $x, y \in G$, and $z \in S$, we have

$$(x,y)z = z(z^{-1}xz, z^{-1}yz)$$

But this is simply a consequence of the following identity, which is easily checked:

$$c(a,b)c^{-1} = (cac^{-1}, cbc^{-1}) \tag{2.10}$$

For the third part, we first note that $(x,y)^n \in Z$. Indeed, the group G/Z is finite of order n, and so the (x,y) considered as an element of G/Z must have order divisible by n. It follows, therefore, that $(x,y)^n$ is trivial in this quotient group, which is to say $(x,y)^n \in Z$. Now we can write

$$\begin{aligned} (x,y)^{n+1} &= (x,y)^n y^{-1} y (x,y) \\ &= y^{-1}(x,y)^n y(x,y) \\ &= y^{-1}(x,y)^{n-1}(x,y)y(x,y) \\ &= y^{-1}(x,y)^{n-1}(x,y^2)y \\ &= y^{-1}(x,y)^{n-1}yy^{-1}(x,y^2)y \\ &= (y^{-1}(x,y)y)^{n-1}y^{-1}(x,y^2)y \\ &= (y^{-1}xy, y)^{n-1}(y^{-1}xy, y^2) \end{aligned}$$

where the last equality uses (2.10) again. Since this is the product of n commutators, this completes the proof. $\square$

Lemma 2.2.1.2. *Let A, B be normal subgroups of G, and suppose the set*

$$S = \{(x,y) \mid x \in A, y \in B\}$$

is finite. Then (A, B) is a finite group.

Proof. First of all, note that we can assume that $G = AB$. Otherwise, we can simply replace G with the subgroup AB. Now, since A and B are normal, it is easy to show that $C = (A, B)$ is normal in G, and hence G acts on the set S via inner automorphisms. That is, G permutes the set S, so we can define a homomorphism

$$\begin{aligned} \alpha : G &\longrightarrow \mathfrak{S}_n \\ g &\mapsto \sigma \end{aligned}$$

where $|S| = n$ and $g.S = \sigma(S)$. Now, since S_n is finite, we have $H = \ker \alpha$ is of finite index in G. Moreover, $H \subset Z_G(C)$. From this, we have that the subgroup $H \cap C$ lies in $Z(C)$, and is of finite index in C. This implies that $Z(C)$ is of finite index in C, and so by Lemma 2.2.1.1 we have that (C, C) is finite. Note now that it suffices to show that $C/(C, C)$ is finite in $G/(C, C)$, since this implies that C is finite in G. Therefore, we can replace G by $G/(C, C)$, which is to say that we can assume that C is abelian, and in particular that S is commmutative.

Now note that, for $a \in A, b \in B$, we have $aba^{-1} \in B$, since B is normal, so $(a, b) = b'b \in B$. Therefore, $C \subset B$, and so, for $c \in C$, commutators (a, c) lie in S, and hence commute with each other. Moreover, we can exploit the commutativity of C to derive the following, for arbitrary $a \in A, c \in C$:

$$\begin{aligned} (a, c)^2 = (aca^{-1}c^{-1})(aca^{-1}c^{-1}) &= (aca^{-1})c^{-1}(aca^{-1}c^{-1}) \\ &= (aca^{-1})(aca^{-1}c^{-1})c^{-1} = ac^2a^{-1}c^{-2} = (a, c^2) \end{aligned}$$

That is, the square of an arbitrary commutator (a, c) can be rewritten as (a, c'). Combining this with the fact that such elements are commutative, and that there are only finitely many such elements since S is finite, we see that (A, C) is necessarily finite. We can therefore once more reduce our problem, by replacing G with $G/(A, C)$, which gives us that $A \subset Z_G(C)$.

Finally then, suppose that $a \in A, b \in B$. Then, since $ba^{-1}b^{-1} \in A$ by normality of A, we have

$$\begin{aligned} (a, b)^2 = (aba^{-1}b^{-1})(aba^{-1}b^{-1}) &= a(ba^{-1}b^{-1})(aba^{-1}b^{-1}) \\ &= a(aba^{-1}b^{-1})(ba^{-1}b^{-1}) = a^2b^{-1}a^{-2}b^{-1} = (a^2, b) \end{aligned}$$

that is, the square of an arbitrary element of S is another element of S. Moreover, since S is commutative, this tells us that (A, B) is finite, as required. □

Proposition 2.2.1.3. *Let A, B be closed subgroups of an algebraic group G.*

1. *If A is connected, then (A, B) is closed and connected.*

2. *If A, B are normal in G, then (A, B) is closed and normal in G. In particular, (G, G) is always closed.*

Proof. For (1), let $y \in B$, and define a morphism

$$\begin{aligned} \gamma_y : A &\longrightarrow G \\ x &\longmapsto xyx^{-1}y^{-1} \end{aligned}$$

Certainly $\gamma_y(e) = e$ for all $y \in B$, and so we use Proposition 1.3.2.4 to note that $(A, B) = \{\gamma_y(A) \mid y \in B\}$ is closed and connected.

For (2), note first that (1) tells us that (A°, B) and (A, B°) are closed and connected subgroups of G, and therefore, it follows from Corollary 1.3.1.10 that their product $C = (A^\circ, B) \cdot (A, B^\circ)$ is, too. Now it suffices to show that $[(A, B) : C]$ is finite, since this will imply that (A, B) is a finite union of closed sets. This is purely a group-theoretic problem, for which we will apply Lemma 2.2.1.2.

In fact, we will consider the group $G' = G/C$, so let A' and B' be the images under this quotient of A and B respectively. We aim to show that $S = \{(x, y) \mid x \in A', y \in B'\}$ is finite. Now we wish to show that each coset xA' contributes only one element to S, that is, given $x_1 \in x_2A^\circ$, then, in G', we have $(x_1, y) = (x_2, y)$ for arbitrary $y \in B'$. Indeed, since $x_2^{-1}x_1 \in A^\circ$, we have, in G, that

$$(x_2^{-1}x_1, y) \in (A^\circ, B) \subset C$$

and so, back in G',

$$\begin{aligned} (x_2^{-1}x_1, y) &= e \\ x_2^{-1}x_1yx_1^{-1}x_2y^{-1} &= e \\ x_1yx_1^{-1} &= x_2yx_2^{-1} \\ x_1yx_1^{-1}y^{-1} &= x_2yx_2^{-1}y^{-1} \\ (x_1, y) &= (x_2, y) \end{aligned}$$

A similar argument shows that each coset yB° contributes only one element (x, y) to S for each $x \in A$. This, combined with the fact that $[A : A^\circ]$ and $[B : B^\circ]$ are both finite, we have that S is finite. Now we can apply Lemma 2.2.1.2 to tell us that $(A', B') = (A, B)/C$ is finite, which gives us that $[(A, B) : C]$ is finite, as required. □

2.2.2 Solvable Groups

Definition 2.2.2.1. An abstract group G is *solvable* if its *derived series* terminates in e, this series being defined inductively as

$$\begin{aligned} \mathscr{D}^0G &= G \\ \mathscr{D}^{i+1}G &= (\mathscr{D}^iG, \mathscr{D}^iG), \quad i \geq 0 \end{aligned}$$

Note that if G is an algebraic group, Proposition 2.2.1.3 tells us that $\mathscr{D}^iG$ is closed and normal for all i, and is connected if G is.

Proposition 2.2.2.2. *Let G be a connected, solvable algebraic group of positive dimension. Then*

$$\dim(G, G) < \dim G$$

Proof. If H', H are connected algebraic groups, with $H' \subsetneq H$, then $\dim H' < \dim H$. Certainly, $(G, G) \subsetneq G$ since G is solvable and nontrivial, and so the result follows immediately. $\square$

Definition 2.2.2.3. The *derived length* of a solvable group G is the least integer d such that $\mathscr{D}^d = \{e\}$.

Proposition 2.2.2.4. *An algebraic group G is solvable if and only if there exists a chain of closed subgroups*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$$

such that $(G_i, G_i) \subset G_{i+1}$ for all $0 \leq i \leq n$.

Proof. Supposing G is solvable, if we set $G_i = \mathscr{D}^i G$, then we get a chain of the required form. So we need to show the contrary, namely that this chain condition is necessary for G to be solvable, so suppose we have such a chain $G_0, \ldots, G_n$. We aim to show that $\mathscr{D}^n G \subset G_n = \{e\}$. We do so by showing, using induction, that $\mathscr{D}^i G \subset G_i$ for each i. Indeed, since $G_0 = \mathscr{D}^0 G = G$, this is trivially true for $i = 0$. Moreover, since clearly $A \subset B$ implies $(A, A) \subset (B, B)$, we have

$$\mathscr{D}^{i+1} G = (\mathscr{D}^i G, \mathscr{D}^i G) \subset (G_i, G_i) \subset G_{i+1}.$$

In particular, $\mathscr{D}^n G \subset G_n = \{e\}$ and so G is solvable. $\square$

Lemma 2.2.2.5.

1. *Subgroups and homomorphic images of a solvable group are again solvable.*
2. *If N is a normal solvable subgroup of G such that G/N is solvable, then G itself is solvable.*
3. *If A, B are normal solvable subgroups of G, then AB is, too.*

Proof. For (1), simply note that if $H \subset G$ is a subgroup, then $\mathscr{D}^i H \subset \mathscr{D}^i G$ for all i. Moreover, given a group homomorphism $\varphi : G \to G'$, it is easy to check that $\mathscr{D}^i \varphi G = \varphi(\mathscr{D}^i G)$ for all i.

For (2), consider the quotient morphism $\alpha : G \to G/N$. We know that, for some n,

$$\{e\} = \mathscr{D}^n (G/N) = \mathscr{D}^n \alpha G = \alpha(\mathscr{D}^n G)$$

and so $\mathscr{D}^n G \subset \ker \alpha = N$. But N is solvable, so, for some m, we have

$$\{e\} = \mathscr{D}^m N \supset \mathscr{D}^{n+m} G$$

so G is solvable.

For (3), note first that $A \lhd AB$ and $B \lhd AB/A$, where B, here means (by abuse of notation), the image of B in the quotient, which itself is solvable by (1). Furthermore, $(AB/A)/B = \{e\}$ and so, applying (2), we get AB/A solvable, and, applying that result again, we get AB solvable. $\square$

Example 17. Let $T = T(n, K)$, the set of invertible upper triangular matrices, and $U = U(n, K)$, the set of invertible upper triangular matrices with diagonal entries equal to 1. Then $U = (T, T)$. Indeed, since for $t, t' \in T$, the diagonal entries of tt' are simply the product of the respective diagonal entries of t and t', then certainly $(T, T) \subset U$. For the reverse inclusion, recall the fact that we used in the proof of Proposition 1.3.2.8, namely that U is generated as a group by matrices

$$u_{ij}(a) = 1 + ae_{ij}$$

where $i < j$ and $a \in K$. That is, every element of U can be written as a product of u_{ij}'s. Suppose now that $t \in D(n, K) \subset T$ with 2 in the i^{th} entry and 1 in the others. Then

$$tu_{ij}(a)t^{-1} = u_{ij}(2a)$$

and so

$$(t, u_{ij}(a)) = u_{ij}(a)$$

Therefore, $U \subset (T, T)$. We have thus shown that $U = (T, T)$.

Moreover, U is solvable, a fact which we now show by using Proposition 2.2.2.4. Consider first the set $\mathfrak{t}(n, K)$ of all upper triangular $n \times n$ matrices, which is an associative subalgebra of $\mathrm{Mat}(n, K)$. Define $\mathfrak{n}$ to be the subset consisting of matrices with 0 on the diagonal. Given elements $\mathsf{x} \in \mathfrak{t}(n, K), \mathsf{y} \in \mathfrak{n}$,

$$(\mathsf{xy})_{ii} = \sum_{k=1}^{n} \mathsf{x}_{ik}\mathsf{y}_{ki} = 0$$

since $\mathsf{x}_{ik} = 0$ if $i > k$, whereas $\mathsf{y}_{ki} = 0$ if $k \geq i$. It follows that $\mathsf{xy} \in \mathfrak{n}$. Similarly,

$$(\mathsf{yx})_{ii} = \sum_{k=1}^{n} \mathsf{y}_{ik}\mathsf{x}_{ki} = 0$$

and so $\mathsf{yx} \in \mathfrak{n}$, and so $\mathfrak{n}$ is a two sided ideal of $\mathfrak{t}(n, K)$. For each integer $h \geq 0$, define the set

$$\mathfrak{n}^h = \{n_1 n_2 \cdots n_h \,|\, n_i \in \mathfrak{n}\}$$

A similar argument to that above shows that each $\mathfrak{n}^h$ is a two sided ideal of $\mathfrak{t}(n, K)$. Furthermore, one can use an induction argument to show that $\mathfrak{n}^h$ is the linear span of all e_{ij} for which $j - i \geq h$. From this we can readily deduce that $U = 1 + \mathfrak{n}$, and, moreover, if we define $U_h = 1 + \mathfrak{n}^h$, we have that U_h is a subgroup of U, which is normal, since $\mathfrak{n}^h$ is a two sided ideal, and is closed since it is defined by polynomial conditions.

Suppose now that $x = 1 + \mathsf{x} \in U_h$ and $y = 1 + \mathsf{y} \in U_l$. Write $x^{-1} = (1 + \mathsf{x}'), y^{-1} = (1 + \mathsf{y}')$, and since U_h, U_l are subgroups, it follows that $\mathsf{x}' \in U_h, \mathsf{y}' \in U_l$. Now,

$$\begin{aligned} xyx^{-1}y^{-1} &= (1 + \mathsf{x})(1 + \mathsf{y})(1 + \mathsf{x}')(1 + \mathsf{y}') \\ &= 1 + (\mathsf{x} + \mathsf{x}') + (\mathsf{y} + \mathsf{y}') + (\mathsf{x} + \mathsf{y})(\mathsf{x}' + \mathsf{y}') \\ &\qquad + (\mathsf{x} + \mathsf{y})\mathsf{x}'\mathsf{y}' + \mathsf{xy}(\mathsf{x}' + \mathsf{y}') + \mathsf{xy} + \mathsf{xyx}'\mathsf{y}' \end{aligned}$$

It is easy to show that $\mathsf{x} + \mathsf{x}' = -\mathsf{x}\mathsf{x}'$, and similarly for $\mathsf{y} + \mathsf{y}' = -\mathsf{y}\mathsf{y}'$, so

$$\begin{aligned} xyx^{-1}y^{-1} &= 1 - (\mathsf{x}\mathsf{x}') - (\mathsf{y}\mathsf{y}') + (\mathsf{x}+\mathsf{y})(\mathsf{x}'+\mathsf{y}') \\ &\quad + (\mathsf{x}+\mathsf{y})\mathsf{x}'\mathsf{y}' + \mathsf{x}\mathsf{y}(\mathsf{x}'+\mathsf{y}') + \mathsf{x}\mathsf{y} + \mathsf{x}\mathsf{y}\mathsf{x}'\mathsf{y}' \\ &= 1 + \mathsf{x}\mathsf{y}' + \mathsf{y}\mathsf{x}' + (\mathsf{x}+\mathsf{y})\mathsf{x}'\mathsf{y}' + \mathsf{x}\mathsf{y}(\mathsf{x}'+\mathsf{y}') + \mathsf{x}\mathsf{y} + \mathsf{x}\mathsf{y}\mathsf{x}'\mathsf{y}' \quad (2.11) \end{aligned}$$

Now, certainly $\mathsf{x}\mathsf{y}, \mathsf{x}'\mathsf{y}', \mathsf{x}\mathsf{y}', \mathsf{x}'\mathsf{y} \in \mathfrak{n}^{h+l}$ by definition, and since $\mathfrak{n}^{h+l}$ is a two sided ideal, equation (2.11) gives us that $xyx^{-1}y^{-1} - 1 \in \mathfrak{n}^{h+l}$, which in turn tells us $(U_h, U_l) \subset U_{h+l}$. Clearly $U_n = \{e\}$, and so U is solvable, and hence so is T.

2.2.3 Nilpotent Groups

Definition 2.2.3.1. The *descending central series* of an abstract group G is defined inductively by

$$\begin{aligned} \mathscr{C}^0 G &= G \\ \mathscr{C}^{i+1} G &= (G, \mathscr{C}^i G) \end{aligned}$$

for all $i \geq 0$.

Proposition 2.2.3.2. *If G is an algebraic group, then $\mathscr{C}^i G$ is closed and normal in G for all i.*

Proof. We prove this by induction. Since $\mathscr{C}^0 = G$, it is trivially true for $i = 0$. Moreover, since $\mathscr{C}^{i+1} = (G, \mathscr{C}^i G)$, we can apply Proposition 2.2.1.3(1) to see that $\mathscr{C}^{i+1} G$ is closed and normal. □

Definition 2.2.3.3. If there exists an integer n such that $\mathscr{C}^n G = \{e\}$, then G is called *nilpotent*. Call the minimum such n the *length* $l(G)$ of the nilpotent group.

The following are results about abstract groups G.

Proposition 2.2.3.4. *If a group G is nilpotent, then it is solvable.*

Proof. This follows from the fact, which we show by induction, that $\mathscr{D}^i G \subset \mathscr{C}^i G$ for all i. This is clearly true for $i = 0$, since $\mathscr{D}^0 G = \mathscr{C}^0 G = G$. Now, since $\mathscr{D}^i G \subset G$, we have

$$\mathscr{D}^{i+1} = (\mathscr{D}^i G, \mathscr{D}^i G) \subset (G, \mathscr{C}^i G) = \mathscr{C}^{i+1}$$

where the last inclusion is by the induction hypothesis. So $\mathscr{D}^i G \subset \mathscr{C}^i G$ and therefore if $\mathscr{C}^n = \{e\}$, then $\mathscr{D}^n = \{e\}$ and so G is solvable. □

Lemma 2.2.3.5.

1. *Subgroups and homomorphic images of a nilpotent group are again nilpotent.*
2. *If $G/Z(G)$ is nilpotent, then G itself is nilpotent.*

3. *Suppose G is nilpotent. If n is the maximum index such that $\mathscr{C}^n G \neq \{e\}$, then*

$$\mathscr{C}^n G = Z(G)$$

In particular, if $G \neq \{e\}$, then $Z(G) \neq \{e\}$.

4. *If H is a proper subgroup of G, with G nilpotent, then H is properly included in $N_G(H)$.*

Proof. For (1) we use use the simple fact that given subgroups $A, B, C, D \subset G$, with $A \subset B$ and $C \subset D$, then $(A, C) \subset (B, D)$. Now, if $H \subset G$, then $\mathscr{C}^0 H \subset \mathscr{C}^0 G$, and, arguing by induction,

$$\mathscr{C}^{i+1} H = (H, \mathscr{C}^i H) \subset (G, \mathscr{C}^i G) = \mathscr{C}^{i+1} G$$

Next we observe that, given $\varphi : G \to G'$, we have $\varphi(G, \mathscr{C}^i G) = (\varphi G, \mathscr{C}^i \varphi G)$, a fact which again can be easily proven by induction. This tells us that homomorphic images of nilpotent groups are again nilpotent.

For (2), consider the quotient homomorphism $\alpha : G \to G/Z(G)$. We know from above that, for some n,

$$\{e\} = \mathscr{C}^n(G/Z(G)) = \mathscr{C}^n(\alpha G) = \alpha(\mathscr{C}^n G)$$

and so $\mathscr{C}^n G \subset \ker \alpha = Z(G)$. But, since $A \subset B$ implies $\mathscr{C}^i A \subset \mathscr{C}^i B$, we have

$$\mathscr{C}^{n+1} G \subset \mathscr{C}^1 Z(G) = \{e\}$$

where the last equality is by the commutativity of $Z(G)$. This shows that G is nilpotent.

For (3), let $y \notin \mathscr{C}^n G$. Then there exists an element $x \in G$ such that $e \neq xyx^{-1}y^{-1} = (x, y)$, and so $y \notin Z(G)$. On the other hand, suppose $y \in \mathscr{C}^n G$. Then, for all $x \in G$, we have $(x, y) \in \mathscr{C}^{n+1} G = \{e\}$ and so $xy = yx$ which means $y \in Z(G)$ as required.

For (4), write $Z = Z(G)$, and suppose firstly that there exists an element $x \in Z - H$. Then $H \subsetneq ZH$, and since $ZH \subset N_G(H)$, we have $H \subsetneq N_G(H)$. Suppose, on the other hand, that $Z \subset H$. We prove the statement by induction on the length $l(G) = n$ of the nilpotent group G. Now, since $H \subsetneq G$, we cannot have $n = 0$ so we start the induction with the case $n = 1$. In this case $G = Z(G)$ so is abelian, which tells us $H \subsetneq G = N_G(H)$. Now suppose that the result is true for lengths less than n, and consider the quotient morphism $\alpha : G \to G/Z$. We claim that $l(G/N) < l(G)$. Indeed, we know that

$$\mathscr{C}^i(G/Z) = \mathscr{C}^i(\alpha G) = \alpha(\mathscr{C}^i G)$$

where the first equality is since α is surjective, and the second by (3). In particular, if $l(G) = n$, we have

$$\mathscr{C}^{n-1}(G/Z) = \mathscr{C}^{n-1}(\alpha G) = \alpha(\mathscr{C}^{n-1} G) = \alpha Z = \{e\}$$

so $l(G/Z) \leq n - 1$. Therefore we can apply the induction hypothesis to give us $H/Z \subsetneq N_{G/Z}(H/Z)$, but this in turn implies that $H \subsetneq N_G(H)$, as required. $\square$

Example 18. The calculations in Example 17 actually indicate that $U(n, K)$ is nilpotent. Indeed, using the notation of Example 17, $U(n, K) = U_1$, and $(U_1, U_h) \subset U_{1+h}$. Since $U_n = \{e\}$, it follows that $U(n, K)$ is nilpotent.

Another set of results will prove useful later:

Proposition 2.2.3.6. *Let G be a connected nilpotent algebraic group of positive dimension. Then*

1. *$Z(G)$ has positive dimension.*
2. *If H is a proper, closed, connected subgroup of G, then*

$$\dim H < \dim N_G(H).$$

In particular, if $\mathrm{codim} H = 1$, *then H is normal in G.*

Proof. For (1), note first that by Proposition 2.2.1.3 (1), we have that $\mathscr{C}^i G$ is connected for all i if G is. In particular, for some n, $\mathscr{C}^n = Z(G) = Z(G)^\circ$, by Lemma 2.2.3.5 (3). Indeed, we see that $Z(G)^\circ \neq \{e\}$ since G has positive dimension, and therefore cannot be the trivial group.

For (2) we argue by an induction similar to that of the proof of Lemma 2.2.3.5 (4), so again write $Z = Z(G)$. Suppose that Z is not included in H, that is, there exists an element $x \in Z - H$. Then $H \subsetneq ZH$, and, moreover, we know that ZH is connected, since both Z and H are and the fact that this former group is the image of a product morphism. It follows that $\dim H < \dim ZH$. But $ZH \subset N_G(H)$, and so $\dim H < \dim N_G(H)$. On the other hand, suppose that $Z \subset H$, and again we consider the inclusion $H/Z \subsetneq G/Z$. We note that G/Z is connected and nilpotent, being the homomorphic image of G which is connected and nilpotent. Moreover, by (1) we know that $\dim G/Z < \dim G$, so we can apply the induction hypothesis to tell us that $\dim(H/Z) < \dim N_{G/Z}(H/Z)$, which is enough to tell us that $\dim H < \dim N_G(H)$. □

2.2.4 Unipotent Subgroups

Definition 2.2.4.1. A subgroup of an algebraic group is called *unipotent* if all its elements are unipotent.

Example 19. $U(n, K)$ is unipotent in $GL(n, K)$.

Theorem 2.2.4.2. *Let G be a unipotent subgroup of $GL(V)$, where V is a nontrivial finite dimensional vector space. Then G has a common eigenvector in V, that is, a nonzero vector v such that $g.v = v$ for all $g \in G$.*

Proof. The proof uses on an application of Burnside's Theorem, which is Corallary 3.5 in Chapter XVII, §3 of [8], namely that, given a subalgebra R of $\mathrm{End}(V)$, with V finite dimensional, which acts irreducibly on V, then $R = \mathrm{End}(V)$.

First we reduce to the case in which V is an irreducible G-module. For suppose there exists a proper, nonzero subspace $W \subset V$ such that $G.W \subset W$. Then G can be considered as a unipotent subgroup of $GL(W)$, and so any vector

$w \in W$ which is fixed by all G is also a vector in V which is a fixed by all G. So we can assume that V is irreducible.

Now, a matrix x is unipotent if and only if $x - 1$ has 0 as its only eigenvalue. By the familiar property involving traces and determinants, this means that for all $x \in G \subset GL(V)$, we have $\mathrm{tr}(x-1) = 0$, whence $\mathrm{tr}(x) = \mathrm{tr}(1) = \dim V$. If we also write $x = 1 + n$ for n nilpotent, then for any other $y \in G$, we have $xy \in G$ and so

$$\mathrm{tr}(y) = \dim V = \mathrm{tr}(xy) = \mathrm{tr}((1+n)y) = \mathrm{tr}(y) + \mathrm{tr}(ny)$$

and so $\mathrm{tr}(ny) = 0$. Note now that if a is a K-linear combination of elements of G, $a = \sum \mu_i y_i$, say, with $\mu_i \in K, y_i \in G$, then for all $x = 1 + n \in G$ we have

$$\mathrm{tr}(na) = \mathrm{tr}\big(n(\sum \mu_i y_i)\big) = \sum \mu_i \mathrm{tr}(ny_i) = 0$$

Now the elements $a \in K[G]$ form a subalgebra of $\mathrm{End}(V)$ which acts irreducibly on V, since G does. Burnside's theorem then tells us that $K[G] = \mathrm{End}(V)$, and, so for all $x = n + 1 \in G$ and all $y \in \mathrm{End}(V)$ we have $\mathrm{tr}(ny) = 0$. By letting y vary over the standard matrices e_{ij}, we see that this implies all the entries of n must be 0, which in turn gives us that $x = 1$ is the only element of G. But $\mathrm{End}(V) = K[G] = K$, which tells us that $V = K$, or $\dim V = 1$, in which case certainly G has a common eigenvector in V. □

Corollary 2.2.4.3. *If G is unipotent subgroup of $GL(n,K)$, then there exists an element $x \in GL(n,K)$ such that*

$$xGx^{-1} \subset U(n,K).$$

In particular, G is nilpotent.

Proof. It suffices to show that $V = K^n$ has a full flag of subspaces stable under the action of G, since in this case, it is immediate that we can find an element $x \in GL(n,K)$ such that $xGx^{-1} \subset T(n,K)$. But since the elements of G are unipotent, they each have 1 as their sole eigenvalue, and hence the diagonal entries of each element of xGx^{-1} must be equal to 1. Hence $xGx^{-1} \subset U(n,T)$.

We use induction on n to prove that $V = K^n$ has a full flag of subspaces stable under G. The case of $n = 1$ is trivial, so we assume the result holds for dimension less than n. We now implement Theorem 2.2.4.2 to give us a common eigenvector $v_1 \in V$ of G. Defining $V_1 = Kv_1$, we see that G stabilises V_1, and so acts on V/V_1. Moreover, the image of G in $GL(V/V_1)$ is again unipotent since morphisms map unipotent elements to unipotent elements. Now V/V_1 has a full flag of subspaces stable under the action of G by the induction hypothesis, and so we only need to append V to the end of this to get the required full flag of V. □

Corollary 2.2.4.3 combined with Example 18 and Proposition 2.2.3.4 tells us:

$$G \text{ unipotent } \Rightarrow G \text{ nilpotent } \Rightarrow G \text{ solvable.}$$

Corollary 2.2.4.4. *If G is a unipotent algebraic group, then $\mathscr{L}(G)$ consists of nilpotent elements.*

Proof. By Theorem 1.4.4.8, we can construct a morphism $\pi : G \longrightarrow GL(n, K)$, such that $G \cong \pi(G)$, where this latter variety is a closed subgroup of $GL(n, K)$. Since morphisms send unipotent elements to unipotent elements, $\pi(G)$ is a unipotent subgroup of $GL(n, K)$ and so by Corollary 2.2.4.3, there exists an element $x \in GL(n, K)$ such that $x\pi(G)x^{-1} = H \subset U(n, K)$.

So $\mathscr{L}(H) \subset \mathscr{L}(U(n, K))$. Since $U(n, K)$ is defined as the zero set in $\mathrm{Mat}(n, K)$ of the linear polynomials $T_{ii} - 1$ and T_{ij} where $i > j$, its Lie algebra is easy to compute, and $\mathscr{L}(U(n, K))$ consists of matrices with zeros on or below the diagonal. Such matrices are evidently nilpotent, and so $\mathscr{L}(H)$ consists of nilpotent elements. But $G \cong H$ by construction, and so $\mathscr{L}(G)$ consists of nilpotent elements, as required. □

2.2.5 The Lie-Kolchin Theorem

Theorem 2.2.5.1 (Lie-Kolchin Theorem). *Let G be a connected, solvable subgroup of $GL(V)$, with V a nontrivial, finite dimensional vector space. Then G has a common eigenvector in V.*

The proof of this theorem is simplified by the following lemma:

Lemma 2.2.5.2. *Let G be an algebraic group, with H a subgroup, not necessarily closed. If H is commutative (respectively solvable), then the same is true of $\bar{H}$.*

Proof. Suppose firstly that H is commutative, and let $x \in H$. Consider the morphism of varieties

$$\begin{aligned} \gamma_x : \bar{H} &\longrightarrow \bar{H} \\ y &\longmapsto yxy^{-1}x^{-1} \end{aligned}$$

γ_x is certainly continuous, and since the singleton $\{e\}$ is closed in the Zariski topology on $\bar{H}$, it follows that $(\gamma_x)^{-1}\{e\}$ is a closed subset of $\bar{H}$. On the other hand, since H is commutative, it follows that $H \subset (\gamma_x)^{-1}\{e\}$, and so $\bar{H} = (\gamma_x)^{-1}\{e\}$. We have shown, then, that elements of H commute with elements of $\bar{H}$, that is, if $x \in H$ then $xyx^{-1} = y$ for all $y \in \bar{H}$.

Now consider an arbitrary element $z \in \bar{H}$, and the morphism

$$\begin{aligned} \gamma_z : \bar{H} &\longrightarrow \bar{H} \\ y &\longmapsto yzy^{-1}z^{-1} \end{aligned}$$

Again, the preimage $(\gamma_z)^{-1}\{e\}$ is closed. We showed above that elements of H commute with elements of $\bar{H}$, and therefore $H \subset (\gamma_z)^{-1}\{e\}$. Together, these facts give $(\gamma_z)^{-1}\{e\} = \bar{H}$, which is to say $\bar{H}$ is commutative, as required.

Suppose now that H is solvable. Then by Proposition 2.2.2.4 there is a chain of subgroups closed in H

$$H = H_0 \supset H_1 \supset \cdots \supset H_n = \{e\}$$

such that $(H_i, H_i) \subset H_{i+1}$ for all i. But this also gives a chain

$$\bar{H} = \bar{H}_0 \supset \bar{H}_1 \supset \cdots \supset \bar{H}_n = \{e\}$$

and since (H_i, H_i) is closed in H_i by Proposition 2.2.1.3(2), and therefore in H, and similarly $(\bar{H}_i, \bar{H}_i)$ is closed in $\bar{H}$, we have $(\bar{H}_i, \bar{H}_i) = \overline{(H_i, H_i)} \subset \overline{H}_{i+1}$ and so Proposition 2.2.2.4 again tells us that $\bar{H}$ is solvable.

□

Proof of Theorem 2.2.5.1. We use induction on $n = \dim V$ and d, the derived length of G. Note also that by Lemma 2.2.5.2, we can assume that G is closed. In the case of $n = 1$ the Theorem is trivial, whilst in the case of $d = 1$, that is, when G is commutative, the result is simply a restatement of Proposition 2.1.1.2.

Suppose firstly that V is a reducible representation, with W a nontrivial proper subspace which is stable under G. If we extend a basis of W to a basis of V, then the matrices representing elements of G have the form

$$\begin{pmatrix} \varphi(x) & * \\ 0 & \psi(x) \end{pmatrix} \tag{2.12}$$

where the map

$$\begin{aligned} \varphi : G &\longrightarrow \varphi(G) \subset GL(W) \\ x &\longmapsto \varphi(x) \end{aligned}$$

is a morphism of algebraic groups. Certainly $\varphi(G)$ is connected, since G is, $\varphi(G)$ is closed by. Proposition 1.3.2.1 (2), and is solvable by Proposition 2.2.2.5 (1). Now $\varphi(G) \subset GL(W)$ satisfies the induction hypothesis, since $\dim W < n$, so there is a common eigenvector $w \in W$ for $\varphi(G)$. Certainly $w \in V$, and, moreover, by construction of φ, we have $Gw = \varphi(G)w$, and so w is also a common eigenvector of G in V.

Now we turn to the remaining case, that of G acting irreducibly on V. Set $G' = (G, G)$. Now, Proposition 2.2.1.3 tells us that G' is normal, closed and connected. Of course, G' has derived length $d - 1$, and so by our induction hypothesis, has a common eigenvector in V. Let W be the span of all such vectors. Now we observe that Lemma 1.5.1.4 says that G' acts diagonally on W. Moreover, Lemma 1.5.1.5 says that since G' is a normal subgroup of G, the action of G stabilises W, and so, and so $W = V$ by irreducibility. Now we have shown that G' acts diagonally on V, that is, its elements can be represented as diagonal matrices, and so we deduce that G' is commutative, which, in turn, tells us that $(G', G') = \{e\}$, or $d \leq 2$.

Now, take $y \in G'$, and choose a basis of V such that y is diagonal. For $x \in G$, since $xyx^{-1}y^{-1} \in G'$, this matrix is also diagonal. We left multiply by the diagonal y to get another diagonal matrix xyx^{-1}. This matrix, being conjugate to y, has the same eigenvalues as y. But both matrices are diagonal, and therefore, given the eigenvalues of y, there are only finitely many possibilities for the matrix xyx^{-1}. This tells us that the image of the morphism

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto xyx^{-1} \end{aligned}$$

is finite, and connected since G is. Therefore this image consists only of y itself. That is to say, $xyx^{-1} = y$ for all $x \in G$, or $G' \subset Z(G)$.

In particular, we see that each element $g \in G'$ is a V-module homomorphism, since, if $x \in G, v \in V$, then $g(xv) = x(gv)$. Moreover, V is an irreducible G-module, so we can apply Schur's Lemma, which is Proposition 1.1 in XVIII §1 of [8]. Schur's Lemma tells us that every G-module homomorphism is either 0 or it is invertible. In particular, for each $g \in G'$, we can find an eigenvalue λ of g since K is algebraically close. But then $g - \lambda I$ is not invertible by definition, and so $g = \lambda I$, that is, all the elements of G' are scalar matrices.

But any commutator of matrices has determinant 1, and so all the elements of G' also have determinants equal to 1. Thus, the scalars in G' are n^{th} roots of unity, of which K contains at most n. Therefore G' is finite, but being connected this implies that $G' = \{e\}$. This, in turn, tells us that $d = 1$, and V being an irreducible representation of G', we have $n = 1$, from which the result immediately follows. $\square$

We have an immediate corollary:

Corollary 2.2.5.3. *Let G be a closed, connected, solvable subgroup of $GL(n, K)$. Then there is an element $x \in GL(n, K)$ such that $xGx^{-1} \subset T(n, K)$, the subgroup of upper triangular matrices.*

Proof. As in the proof of Corollary 2.2.4.3, it suffices to show that $V = K^n$ has a full flag of subspaces of V which are stable under G, which we prove by induction. The case of $n = 1$ is trivial, so we assume the result holds for dimension less than n. We use Theorem 2.2.5.1 to give us a common eigenvector for G, namely $v_1 \in V$. We now define the space $V_1 = \operatorname{span}\{v_1\}$, and since G stabilises V_1, it acts on the vector space V/V_1 in a canonical fashion. Moreover, since the image of G in $GL(V/V_1)$ is again connected and solvable, our induction allows us to conclude that we have a full flag of subspaces of V/V_1 which are stable under G. Appending V to the end of this flag gives the required full flag of V, and so G is conjugate to a group of triangular matrices. $\square$

2.3 The Structure of a Solvable Group

We conclude this chapter with a thorough description of the structure of a solvable group. A few technical results are required in the first instance, but from this abstruse technicality we get such important results as Proposition 2.3.1.9, which says that the global and infinitessimal centralisers of a d-group coincide. Finally we are able to split up a solvable group into a maximal torus and a maximal unipotent subgroup.

2.3.1 A Theorem About Centralisers

Theorem 2.3.1.1. *Let U be a connected unipotent subgroup of an algebraic group G, with $s \in G$ a semisimple element which normalises U. Define the usual map on G, $\gamma_s(x) = xsx^{-1}s^{-1}$, and set $M = \gamma_s(U), C = C_U(s)$. Then*

1. *C is a closed subgroup and M is a closed, irreducible subvariety of U.*
2. *The product morphism*
$$\tau : C \times M \to U$$
is bijective
3. *C is connected.*
4. *$\gamma = \gamma_s|_M$ is a bijection of M onto itself.*

We will get to the proof of Theorem 2.3.1.1 later, since it will require a few additonal results. To begin with, we need the following lemma:

Lemma 2.3.1.2. *Let G be an algebraic group, with $x \in G$. Then*
$$\mathscr{L}(C_G(x)) \subset \mathfrak{c}_{\mathfrak{g}}(x).$$
In the case $G = GL(n, K)$,
$$\mathscr{L}(C_G(x)) = \mathfrak{c}_{\mathfrak{g}}(x).$$

Proof. Let $\gamma_x : G \to G$ be the morphism which sends y to $yxy^{-1}x^{-1}$. There is a quick formula to calculate the differantial $d(\gamma_x)_e$ of this map, given in Proposition 10.1 of [6]:
$$d(\gamma_x)_e = 1 - \mathrm{Ad}(x)$$
Since $\ker(\gamma_x) = C_G(x)$, we have $\mathscr{L}(C_G(x)) \subset \ker(d\gamma_x)$ but this latter set is equal to $\mathfrak{c}_{\mathfrak{g}}(x) = \{\mathsf{x} \in \mathfrak{g} \mid \mathrm{Ad}(x)\mathsf{x} = \mathsf{x}\}$. In the case where $G = GL(n, K)$, we can cite another differential formula, namely Lemma 10.3 B of [6]: if $x \in GL(n, K)$ and $\mathsf{x} \in \mathfrak{gl}(n, K)$, then
$$\mathrm{Ad}x(\mathsf{x}) = x\mathsf{x}x^{-1}$$
a fact which we will use often. In this case, then, $\mathsf{x} \in \mathfrak{c}_{\mathfrak{g}}(x)$ if and only if x commutes with x. It follows that
$$\mathscr{L}(C_G(x)) = \{\mathsf{x} \in \mathfrak{c}_{\mathfrak{g}}(x) \mid \det \mathsf{x} \neq 0\}$$
that is, the set defined by the non-vanishing of the polynomial det. Therefore $\mathscr{L}(C_G(x))$ is an open affine set of $\mathfrak{c}_{\mathfrak{g}}(x)$, and in particular $\dim(\mathscr{L}(C_G(x)) = \mathfrak{c}_{\mathfrak{g}}(x)$. Combining this with the inclusion above gives $\mathscr{L}(C_G(x) = \mathfrak{c}_{\mathfrak{g}}(x)$ as required. $\square$

Definition 2.3.1.3. Define the *conjugacy class* of an element $g \in G$, denoted by $Cl_G(g)$, to be the set

$$Cl_G(g) = \{xgx^{-1} \mid x \in G\}$$

It is a fact, whose proof we omit here, that

$$\dim Cl_G(g) = \dim G - \dim C_G(g) \tag{2.13}$$

This can be shown using Theorem 4.3.3 of [6] and the map $\gamma_g : G \to G$. Indeed, just as in the proof of Proposition 1.3.2.1(4), the theorem applied to γ_x says that there exists an element $x \in \gamma_g(G)$ such that

$$\dim G - \dim \gamma_g(G) = \dim \gamma_g^{-1}(x)$$

But $\gamma_g(G) = Cl_G(g)$ and $\gamma_g^{-1}(x) \cong \gamma_g^{-1}(g) = C_G(g)$, and so equation (2.13) follows.

Proposition 2.3.1.4. *Let H be a closed subgroup of $GL(n, K)$ which is normalised by a semisimple element $s \in G$. Then $\mathfrak{c}_{\mathfrak{h}}(s) = \mathscr{L}(C_H(s))$ and the sum of all nontrivial eigenspaces of* Ads *in $\mathfrak{h}$ is the tangent space of $Cl_H(s)s^{-1}$. That is,*

$$\mathfrak{h} = \mathfrak{c}_{\mathfrak{h}}(s) \oplus \mathscr{T}\big(Cl_H(s)s^{-1}\big)_e \tag{2.14}$$

Proof. The first thing to observe is that, since s is semisimple, so is Ads, and therefore we can construct two Ads-eigenspace decompositions:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{c}_{\mathfrak{g}}(s) \oplus \mathfrak{n} \\ \mathfrak{h} &= \mathfrak{c}_{\mathfrak{h}}(s) \oplus \mathfrak{n}' \end{aligned} \tag{2.15}$$

so $\mathfrak{n}, \mathfrak{n}'$ are the sums of the λ-eigenspaces ($\lambda \neq 1$) of Ads in $\mathfrak{g}$ and $\mathfrak{h}$, respectively. In particular, observe that $\mathfrak{n}' = \mathfrak{h} \cap \mathfrak{n}$. The aim of the proof, then, is to show that $\mathfrak{n}' = \mathscr{T}\big(Cl_H(s)s^{-1}\big)_e$.

As additional notation, we define

$$\begin{aligned} M &= Cl_G(s)s^{-1} \\ M' &= Cl_H(s)s^{-1} \end{aligned}$$

By equation (2.13) above, we note that $\dim M = \dim G - \dim C_G(s)$ and $\dim M' = \dim H - \dim C_H(s)$. Moreover, since $e \in M \cap M'$, we can define the tangent spaces of these varieties at the point e. Indeed, since $M \cap G$, then

$$\mathscr{T}(M)_e \subset \mathscr{T}(G)_e = \mathfrak{g}$$

and since $s \in N_G(H)$, we have $M' \subset H$, which gives

$$\mathscr{T}(M')_e \subset \mathscr{T}(H)_e = \mathfrak{h}$$

This equation, combined with the fact that $M' \subset M$, gives

$$\mathscr{T}(M')_e \subset \mathscr{T}(M)_e \cap \mathfrak{h} \tag{2.16}$$

Consider the morphism $\gamma_s : G \to G$, which is defined by $\gamma_s(x) = xsx^{-1}s^{-1}$. Certainly, $M = \gamma_s(G)$. Furthermore, it is a fact that the differential of γ_s is the linear map $1 - \mathrm{Ad}s : \mathfrak{g} \to \mathfrak{g}$. Therefore, $(1 - \mathrm{Ad}s)\mathfrak{g} \subset \mathscr{T}(M)_e$. On the other hand, we have $\ker(d\gamma_s) = \mathfrak{c}_{\mathfrak{g}}(s)$, since $\mathrm{Ad}s(\mathrm{x}) = s\mathrm{x}s^{-1}$. This, combined with (2.15), tells us that $\mathrm{im}(d\gamma_s) = \mathfrak{n}$, and so we have $\mathfrak{n} \subset \mathscr{T}(M)_e$. But since $\mathfrak{c}_{\mathfrak{g}}(s) = \mathscr{L}(C_G(s))$, by Lemma 2.3.1.2, we have

$$\dim \mathscr{T}(M)_e = \dim M = \operatorname{codim} C_G(s) = \operatorname{codim} \mathfrak{c}_{\mathfrak{g}}(s) = \dim \mathfrak{n}$$

and therefore $\mathscr{T}(M)_e = \mathfrak{n}$.

This can now be combined with equation (2.16) to give

$$\mathscr{T}(M')_e \subset \mathscr{T}(M)_e \cap \mathfrak{h} = \mathfrak{n} \cap \mathfrak{h} = \mathfrak{n}' \tag{2.17}$$

This then tells us that

$$\dim H - \dim C_H(s) = \dim M' \leq \dim \mathfrak{n}' = \dim \mathfrak{h} - \dim \mathfrak{c}_{\mathfrak{h}}(s)$$

where the last equality is by equation (2.15). It follows, therefore, that $\dim C_H(s) \geq \dim \mathfrak{c}_{\mathfrak{h}}(s)$, and this, combined with the inclusion given in Lemma 2.3.1.2, implies $\mathscr{L}(C)_H(s) = \mathfrak{c}_{\mathfrak{h}}(s)$, which is the first statement of the proposition. This also means that $\dim \mathscr{T}(M')_e = \dim \mathfrak{n}'$, which, combined with the inclusion in (2.17), proves the second statement of the proposition, and we are done.

□

Proposition 2.3.1.5. *Let H be a closed subgroup of $G = GL(n, K)$ which is normalised by a semisimple element $s \in G$. Then $Cl_H(s)$ is closed.*

Proof. The proof requires us to define a set involving both characteristic and minimal polynomials, so we begin with some notation. For an endomorphism $A \in \mathrm{End}(V)$, and T an indeterminate, write $M_A(T)$ for the minimal polynomial of A, that is, the monic polynomial of least degree such that $M_A(A) = 0$, and write $c_A(T)$ for the characteristic polynomial of A. Note that although these might not be the same, the Cayley-Hamilton theory says that an element $\lambda \in K$ is a root of $M_A(T)$ if and only if it is an eigenvalue of A, and therefore $M_A(T)$ divides $c_A(T)$. Moreover, A is semisimple if and only if $M_A(T)$ has no repeated roots. The

We define a set

$$W = \{x \in N_G(H) \mid M_s(x) = 0, \text{ and } c_{\mathrm{Ad}x|_{\mathfrak{h}}}(T) = c_{\mathrm{Ad}s|_{\mathfrak{h}}}(T)\}$$

Of course, $\mathrm{Ad}x|_{\mathfrak{h}}, \mathrm{Ad}s|_{\mathfrak{h}}$ are both endomorphisms, and so the definition makes good sense. The conditions on W are polynomial, and hence enough to show that W is closed in $N_G(H)$, and since $N_G(H)$ itself is closed, we see that W is closed in G. We also wish to show that W is stable under conjugation by H, that is, for any $h \in H$, $x \in W$, we have $hxh^{-1} \in W$. Certainly $hxh^{-1} \in N_G(H)$. It is not difficult to see that $M_s(hxh^{-1}) = 0$ if $M_s(x) = 0$, since $a_i(hxh^{-1})^i = h(a_ix^i)h^{-1}$, and $\sum_i h(a_ix^i)h^{-1} = h(\sum_i a_ix^i)h^{-1}$. Moreover, since $c_{B^{-1}AB}(T) = c_A(T)$ for any endomorphisms A and B, it follows that $c_{\mathrm{Ad}(hxh^{-1})|_{\mathfrak{h}}}(T) = c_{\mathrm{Ad}x|_{\mathfrak{h}}}(T)$. Therefore, if $x \in W$ and $h \in H$, the second condition of W is also satisfied for hxh^{-1}. We have shown, then, that W is stable under conjugation by H.

Now, the first condition on elements of W says that $M_s(T)$ vanishes at x and therefore has $M_x(T)$ as a factor. But $M_s(T)$ is a product of distinct linear factors since s is semisimple, and thus $M_x(T)$ is also a product of linear factors, from which we conclude that x is semisimple also.

We can now apply Proposition 2.3.1.4. In particular,

$$\dim Cl_H(x) = \mathrm{codim} C_H(x) = \mathrm{codim} \mathfrak{c}_{\mathfrak{h}}(x) = \dim \mathfrak{h} - m_1(x)$$

where $m_1(x)$ denotes the multiplicity of the 1-eigenvalue in $\mathrm{Ad}x|_{\mathfrak{h}}$. Now, the second condition on elements of W forces $m_1(x) = m_1(s)$, which tells us that all conjugacy classes $Cl_H(x)$ for elements in W have the same dimension. These conjugacy classes are the H-orbits of the action on W defined by

$$\begin{aligned} H \times W &\longrightarrow W \\ (h, x) &\longmapsto hxh^{-1} \end{aligned}$$

Since they all have the same dimension, they are all of minimal dimension, and by Proposition 1.4.3.3, orbits of minimal dimension are closed in W, and therefore are closed in G. In particular, $Cl_H(s)$ is closed, as required. □

It is convenient to use the following lemmas in the proof of Theorem 2.3.1.1:

Lemma 2.3.1.6. *With the notation as in Theorem 2.3.1.1, if $x \in U, y \in C$, then*

$$\gamma_s(xy) = \gamma_s(x) = \gamma_s(x)\gamma_s(y)$$

Proof. In the first place,

$$\gamma_s(xy) = xysy^{-1}x^{-1}s^{-1} = xyy^{-1}sx^{-1}s^{-1} = xsx^{-1}s^{-1}$$

while on the other hand,

$$\gamma_s(x) = xsx^{-1}s^{-1} = xsx^{-1}s^{-1}yy^{-1}ss^{-1} = xsx^{-1}s^{-1}ysy^{-1}s^{-1} = \gamma_s(x)\gamma_s(y)$$

as required. □

Lemma 2.3.1.7. *With the notation as in Theorem 2.3.1.1, if $x \in Z(U), y \in U$, then*

$$\gamma_s(xy) = \gamma_s(x)\gamma_s(y)$$

which in turn means $\gamma_s(x^{-1}) = \gamma_s(x)^{-1}$.

Proof. Write

$$\gamma_s(xy) = xysy^{-1}x^{-1}s^{-1} = x(ysy^{-1}s^{-1})x^{-1}(xsx^{-1}s^{-1})$$

Now, since s normalises U by assumption, we have $sy^{-1}s^{-1} \in U$. Now, $x \in Z(U)$, so the above equation gives us

$$\gamma_s(xy) = x(ysy^{-1}s^{-1})x^{-1}(xsx^{-1}s^{-1}) = xx^{-1}(ysy^{-1}s^{-1})(xsx^{-1}s^{-1})$$

or $\gamma_s(xy) = \gamma_s(y)\gamma_s(x)$. The lemma now follows from the fact that $\gamma_s(xy) = \gamma_s(yx)$, since $x \in Z(U)$. $\square$

Proof of Theorem 2.3.1.1. For (1), we apply Proposition 2.3.1.5 to show that $M = Cl_U(s)s^{-1}$ is closed in G and therefore in U, and we showed that $C = C_H(s)$ is closed in Corollary 1.4.1.8. The fact that $M = \gamma_s(U)$ guarantees that M is irreducible, since $U = U^\circ$ by assumption. The fact that M is contained in U is simply because $s \in N_G(U)$.

For (2) we first show, in several steps, that τ is bijective. First we deal with the case when U is commutative, that is, $U = Z(U)$. In this case, Lemma 2.3.1.7 tells us that $\gamma_s|_U$ is a group homomorphism, whose kernel is C, and whose image is M, making M a group. Applying Proposition 1.3.2.1 (4) to $\gamma_s|_U$ tells us that $\dim U = \dim C + \dim M$, which is enough to guarantee that τ is surjective. Since U, M and C are all groups, τ is an algebraic group homomorphism, so to show injectivity of τ, we can examine $\ker \tau$. Consider $x \in C, y \in M$ such that $xy = e$. Now $y = \gamma_s(u)$ for some u, so $e = xy = xusu^{-1}s^{-1}$ or

$$usu^{-1} = x^{-1}s \tag{2.18}$$

Now, $x^{-1} \in C \subset U$ is unipotent, and s is semisimple, by assumption. Moreover, the right hand side of (2.18) commutes, since $x \in C$. It follows that the right hand side of (2.18) is the Jordan decomposition of the left hand side. But this left hand side is a conjugate of s, and so is semisimple. It follows that $x = e$, which in turn forces $y = e$, so τ is injective, as required.

We now move to the case of general U, for which we use induction on $\dim U$. Of course, the result is trivial for $\dim U = 0$, since U is connected by assumption. Now, since U is nilpotent, Proposition 2.2.3.6(1) tells us that $Z(U)$ contains a nontrivial, connected subgroup, $V = Z(U)^\circ$. Of course, if $V = U$, then U is commutative, and we are done. So we are left with the case $V \subsetneq U$, and for our induction we will be considering the group $U' = U/V$, which has smaller dimension than U.

Now we show that $s \in N_G(V)$. Indeed, suppose $v \in V \subset Z(U)$. Then $\mathrm{Int}s(v) = svs^{-1} = u \in U$ since s normalises U. Now suppose $u' \in U$ is some arbitrary element. We will show that $uu' = u'u$, thus proving that $u \in Z(U)$. First we write $u'' = s^{-1}u's \in U$. Then

$$uu' = (svs^{-1})u'(ss^{-1}) = svu''s^{-1} = su''vs^{-1} = ss^{-1}u'svs^{-1} = u'u$$

and so $u \in Z(U)$. We have shown that $\mathrm{Int}s$ maps V into $Z(U)$. But $\mathrm{Int}s(V)$ is connected, since V is, so $\mathrm{Int}s(V) \subset V$, which is to say $s \in N_G(V)$.

Now, write $G' = N_G(V)/V$, and define the quotient homomorphism

$$\pi : N_G(V) \longrightarrow N_G(V)/V = G'$$

and set $s' = \pi(s)$. Note further that $U \subset N_G(V)$ since $V \subset Z(U)$, and so $U' \subset G'$, where $U' = U/V$ as above. Additionally, since $C, M \subset U$, and so $C, M \subset N_G(V)$. Now G', U' and s' satisfy the assumptions of the theorem, and since $\dim U' < \dim U$, we can apply the induction hypothesis. That is, setting $M' = \gamma_{s'}(U')$, the product morphism

$$\tau' : C_{U'}(s') \times M' \longrightarrow U'$$

is bijective. Note further, that $M' = \pi(M)$, since $\pi\gamma_s(U) = \gamma_{s'}\pi(U) = \gamma_{s'}(U')$. We also note here that, since $\dim V < \dim U$, the induction hypothesis also applies to G, V and s, and so the product morphism

$$\tau_V : C_V(s) \times \gamma_s(V) \longrightarrow V$$

is also bijective.

We are now finally ready to show that τ is injective, so suppose elements $z_1, z_2 \in C$ and $x, y \in M$ are such that $\tau(z_1, x) = \tau(z_2, y)$, or $z_1 x = z_2 y$. Of course, C is a group, so by setting $z = z_2^{-1} z_1 \in C$, we can rephrase the expression as $zx = y$. In order to use our induction hypothesis, we apply π to both sides of this last equation, and note that $\pi(x), \pi(y) \in \pi(M) = M'$. Moreover, since

$$\pi(z)s' = \pi(z)\pi(s) = \pi(zs) = \pi(sz) = \pi(s)\pi(z) = s'\pi(z) \qquad (2.19)$$

it follows that $\pi(z) \in C_{U'}(s')$. Altogether, this means

$$\tau'\big(\pi(z), \pi(x)\big) = \pi(z)\pi(x) = \pi(zx) = \pi(y) = \tau'\big(e, \pi(y)\big)$$

Now, τ' is injective by the induction hypothesis, so we have $\pi(z) = e$, or $z \in V$, and $\pi(x) = \pi(y)$. We now write $x, y \in M$ as $\gamma_s(u), \gamma_s(v)$ respectively, whence

$$z(usu^{-1}s^{-1}) = zx = y = vsv^{-1}s^{-1}$$

or $z(usu^{-1}) = vsv^{-1}$. Now, clearly both usu^{-1}, vsv^{-1} are semisimple, and since $z \in V \subset Z(U)$, z is unipotent. Moreover,

$$z(usu^{-1}) = (uz)su^{-1} = us(s^{-1}zs)u^{-1} = us(zu^{-1}) = (usu^{-1})z$$

This tells us that $z(usu^{-1}) = vsv^{-1}$ is the Jordan decomposition of the semisimple right hand side, and therefore $z = e$, whence $x = y$ and $z_1 = z_2$. Therefore τ is injective.

We next have to show that τ is surjective. We first show that $\gamma_s(V) = M \cap V$. The inclusion $\gamma_s(V) \subset M \cap V$ is easy, so for the reverse inclusion, take $v \in M \cap V$. Since $v \in V$ and τ_V is surjective, we have $\tau_V(z, y) = zy = v$ for some $z \in C_V(s), y \in \gamma_s(V)$. But $v \in M$, too, so $v = \tau(e, v)$. Now, we have already shown that τ is injective, so $e = z, v = y \in \gamma_s(V)$, and therefore $M \cap V \subset \gamma_s(V)$. We now have to show another identity, namely $C_{U'}(s') = \pi(C)$. One direction is easy: for $z \in C$, equation (2.19) already told us $\pi(z) \in C_{U'}(s')$, whence $\pi(C) \subset C_{U'}(s')$. For the reverse inclusion, suppose $\pi(x) \in C_{U'}(s')$, that is, $\pi(x)\pi(s) = \pi(s)\pi(x)$. Now clearly $\gamma_{s'}(x) = e$, but this means $\pi(\gamma_s(x)) = \gamma_{s'}(x) = e$, or $\gamma_s(x) \in \ker \pi$. Indeed,

$$\gamma_s(x) \in \ker \pi \cap M = V \cap M = \gamma_s(V)$$

So, for some $v \in V$, we have $\gamma_s(x) = \gamma_s(v)$. But $v \in V \subset Z(U)$, and so Lemma 2.3.1.7 tells us that $e = \gamma(v^{-1})\gamma_s(x) = \gamma_s(v^{-1}x)$, or $v^{-1}x \in C$. Therefore

$$\pi(v^{-1}x) = \pi(v^{-1})\pi(x) = \pi(x) \in \pi(C)$$

and so $C_{U'}(s') \subset \pi(C)$, as required.

Now we can finally move on to showing that τ is surjective. Since τ' is bijective, we have

$$\pi(U) = U' = C_{U'}(s')M' = \pi(C)\pi(M) = \pi(CM)$$

which, by definition of π, means $U = CMV$. But, since $V \subset Z(U)$,

$$U = CMV = CVM$$

Now, τ_V is bijective, so

$$U = CVM = CC_V(s)\gamma_s(V)M \tag{2.20}$$

Since $C_V(s) \subset C$ we have $CC_V(s) = C$. Using Lemma 2.3.1.7 again, $\gamma_s(V)M = M$ since $V \subset Z(U)$. Putting these expressions together with (2.20) means $U = CM$, as required. This concludes the proof of (2).

We now turn to (3). Let $D = \tau(C^\circ \times M)$. Then D is an irreducible subset of U. Now, since C is an algebraic group,

$$C = \bigcup_{i=1}^{m} x_i C^\circ$$

where $x_i \in C$, with $x_0 = e$, and, moreover, this is a disjoint union. It follows that

$$U = \bigcup_{i=1}^{m} x_i D \tag{2.21}$$

is also a disjoint union. Now, since both C° and M are closed, it follows that D is constructible, and indeed, so is each subset x_iD. Constructibility guarantees the existence of open sets $U_i \subset x_iD$ such that $\bar{U_i} = \overline{x_iD}$. Combining this with equation (2.21) gives

$$U \supset \bigcup_{i=1}^{m} U_i$$

where the union is disjoint. But since U is irreducible, the intersection of nonempty proper open sets cannot be empty, which means in fact $\bigcup_{i=1}^{m} U_i = U_i$ for some U_i. In turn, then, $\bigcup_{i=1}^{m} x_iD = x_iD$ for some x_i. Referring to equation (2.21) again tells us that $U = x_iD$, but since $e \in U$, it follows that $U = D$, But τ is bijective, so $C^\circ \times M = \tau^{-1}(U) = C \times M$, from which we can conclude $C^\circ = C$, as required.

The only remaining component of the theorem is (4). Now $M = \gamma_s(U) = \gamma_s(CM)$, since τ is bijective. Moreover, Lemma 2.3.1.6 says that $\gamma_s(CM) = \gamma_s(M)$, so putting these together we have $M = \gamma_s(M)$, that is, γ is surjective. To prove injectivity, suppose $x, y \in M$ are such that $\gamma(x) = \gamma(y)$, that is, $xsx^{-1}s^{-1} = ysy^{-1}s^{-1}$ from which it is easy to conclude that $yx^{-1} \in C = \ker \gamma_s$. Since we have $x, y \in M$ and $e, yx^{-1} \in C$, we can apply τ to get $\tau(e, y) = \tau(yx^{-1}, x)$. The injectivity of τ then guarantees that $x = y$, and so γ is injective, and we are done. $\square$

Another few results about centralisers will be required later:

Proposition 2.3.1.8. *Let T be a torus of G, an algebraic group. Let H be a closed subgroup of G, which is stabilised by* Int$T$. *Then there exists an element $x \in T$ such that $C_H(x) = C_H(T)$ and $\mathfrak{c}_{\mathfrak{h}}(x) = \mathfrak{c}_{\mathfrak{h}}(T)$.*

Proof. By Theorem 1.4.4.8 we may assume, without loss of generality, that $G = GL(V)$ for some V. By Proposition 2.1.5.6, since T is diagonal we can decompose

$$V = \bigoplus_{\chi \in \mathrm{X}(T)} V_\alpha$$

where V_α is the weight space assoctiated to the character $\alpha \in \mathrm{X}(T)$. Only finitely many of them are nonzero, since V is finite dimensional, so label these V_i for $1 \leq i \leq r$, each corresponding to a character $\alpha_i \in \mathrm{X}(T)$. Let $\alpha_i \neq \alpha_j$. Then the morphism of algebraic groups $\alpha_i\alpha_j^{-1}$ which sends an element $t \in T$ to $\alpha_i(t)\alpha_j(t)^{-1}$ cannot be trivial. But T is connected, and so $\mathrm{im}(\alpha_i\alpha_j^{-1})$ is a nontrivial connected subgroup of $\mathbb{G}_m$, which means $\dim \mathrm{im}(\alpha_i\alpha_j^{-1}) = 1$, and so $\mathrm{im}(\alpha_i\alpha_j^{-1}) = \mathbb{G}_m$. In particular, $\dim T = \dim \ker(\alpha_i\alpha_j^{-1}) + \dim \mathrm{im}(\alpha_i\alpha_j^{-1})$ and so $\ker(\alpha_i\alpha_j^{-1})$ is a subtorus of codimension 1 in T.

This fact tells us that $T \neq \bigcap \ker(\alpha_i\alpha_j^{-1})$, since $\dim T > \dim \ker(\alpha_i\alpha_j^{-1})$. It follows that there must exist an element $x \in T$ such that $x \notin \ker(\alpha_i\alpha_j^{-1})$ for any choice of $i \neq j$.

Let $M = GL(V_1) \times \cdots \times GL(V_r)$ be considered as a subgroup of $GL(V) = GL(V_1 \oplus \cdots \oplus V_r)$. Now take an element $h \in M \cap H$. If $v \in V_i$, then $hv \in V_i$, so

$$(x^{-1}hx)v = \alpha_i(x)x^{-1}hv = \alpha_i(x)\alpha_i(x^{-1})hv = hv$$

But this holds for all i, and so $x^{-1}hx = h$. On the other hand, suppose $h \in C_H(x)$. If $v \in V_i$, then write

$$hv = \lambda_1 v_1 + \cdots + \lambda_r v_r \tag{2.22}$$

so

$$x(hv) = \alpha_1(x)\lambda_1 v_1 + \cdots + \alpha_r(x)\lambda_r v_r$$

On the other hand,

$$h(xv) = \alpha_i(x)hv = \alpha_i(x)\lambda_1 v_1 + \cdots + \alpha_i(x)\lambda_r v_r$$

Since $hx = xh$, we have

$$\alpha_i(x)\lambda_j = \alpha_j(x)\lambda_j$$

for all $1 \leq j \leq r$. But x was specifically chosen such that $\alpha_i(x) = \alpha_j(x)$ if and only if $i = j$, which forces $\lambda_j = 0$ if $j \neq i$. Therefore, equation (2.22) becomes $hv = \lambda_i v_i$, and so $h \in M$. Therefore $C_H(x) = H \cap M$.

Suppose now that $h \in H \cap M$, and let $t \in T$ be arbitrary. If $v \in V_i$, then $hv \in V_i$ and so

$$(t^{-1}ht)v = \alpha_i(t)t^{-1}hv = \alpha_i(t)\alpha_i(t^{-1})hv = hv$$

Again this holds for all i, which implies $t^{-1}ht = h$, and therefore $H \cap M \subset C_H(D)$. But we already saw $C_H(x) = H \cap M$, and so $C_H(x) \subset C_H(D)$. The reverse inclusion is obvious, and so we conclude that $C_H(x) = C_H(D)$, as required. The fact that $\mathfrak{c}_{\mathfrak{h}}(x) = \mathfrak{c}_{\mathfrak{h}}(D)$ is proven similarly, in that the intermediate step is to show that both subalgebras are equal to $\mathscr{L}(M) \cap \mathscr{L}(H)$. □

Proposition 2.3.1.9. *Let H be a closed subgroup of an algebraic group G, with H normalised by a d-group D. Then*

$$\mathscr{L}(C_H(D)) = \mathfrak{c}_{\mathfrak{h}}(D)$$

Proof. We use induction on the dimension of H. Of course, if $\dim H = 0$, then the result is trivially true since $\mathscr{L}(C_H(D)) = \mathfrak{c}_{\mathfrak{h}}(D) = \{0\}$. So suppose $\dim H = n$, and that the result holds for groups of dimension less than n.

Suppose, firstly, that for all $s \in D$, we have $\dim H \leq \dim C_H(s)$. This implies $H^\circ \subset C_H(D)$, and so

$$\mathfrak{h} = \mathscr{L}(H^\circ) \subset \mathscr{L}(C_H(D))$$

Therefore $\mathfrak{h} = \mathfrak{c}_{\mathfrak{h}}(D) = \mathscr{L}(C_H(D))$.

Suppose now that there is an element $s \in D$ such that $H' = C_H(s)$ is of dimension less than n. Now, if $h \in H', t \in D$, then

$$s(tht^{-1})s^{-1} = t(shs^{-1})t^{-1} = tht^{-1}$$

where the first equality holds since D is commutative. This shows us that $tht^{-1} \in H'$, or that $D \subset N_G(H')$. We can therefore apply the induction hypothesis, to give us $\mathscr{L}(C_{H'}(D)) = \mathfrak{c}_{\mathfrak{h}'}(D)$.

Certainly $C_H(D) \subset C_H(s) = H'$, and so $C_{H'}(D) = C_H(D) \cap H' = C_H(D)$. Furthermore, by Proposision 2.3.1.4, $\mathfrak{c}_{\mathfrak{h}}(s) = \mathscr{L}(C_H(s)) = \mathfrak{h}'$. But $\mathfrak{c}_{\mathfrak{h}}(D) \subset \mathfrak{c}_{\mathfrak{h}}(s)$ since $s \in D$, and so $\mathfrak{c}_{\mathfrak{h}}(D) \subset \mathfrak{h}'$. Putting all this together, we get

$$\mathscr{L}(C_H(D)) = \mathscr{L}(C_{H'}(D)) = \mathfrak{c}_{\mathfrak{h}'}(D) = \mathfrak{c}_{\mathfrak{h}}(D) \cap \mathfrak{h}' = \mathfrak{c}_{\mathfrak{h}}(D)$$

and we are done. □

Corollary 2.3.1.10. *Suppose a d-group D acts on an algebraic group H, then*

$$\mathscr{L}(H)^D = \mathscr{L}(H^D)$$

Proof. Set $G = D \ltimes H$. Then G is an algebraic group with subgroups D, H such that $D \subset N_G(H)$. In this case, $H^D = C_H(D)$. Moreover, since D acts on $\mathscr{L}(H)$ via Ad, we have $\mathfrak{c}_{\mathfrak{h}}(D) = \mathfrak{h}^D$. Applying Proposition 2.3.1.9 then gives

$$\mathfrak{h}^D = \mathfrak{c}_{\mathfrak{h}}(D) = \mathscr{L}(C_H(D)) = \mathscr{L}(H^D)$$

as required. □

Corollary 2.3.1.11. *Let a d-group D act on algebraic groups G, G', and let $\varphi : G \to G'$ be an epimorphism which is equivariant with respect to the action of D. Then φ maps the identity component of G^D to that of G'^D.*

Proof. Let $K = \ker\varphi$ and consider the factorisation of φ through the quotient q:

$$G \xrightarrow{q} G/K \xrightarrow{\varphi'} G'$$

where $\varphi'(gK) = \varphi(g)$. Then φ' is bijective. Moreover, D clearly acts on the group G/K, since K is D-invariant. That action is given by $t.gK = (t.g)K$ where $t \in D$ and $g \in G$, and therefore

$$t\varphi'(gK) = t\varphi(g) = \varphi(tg) = \varphi'((tg)K)$$

and so the map φ' is in fact D-equivariant. This means we may as well assume that $G' = G/K$. Then, by Theorem 3, the map differential map $\mathfrak{g} \to \mathfrak{g}'$ is also surjective, and, indeed, that $\mathscr{L}(K) = \ker d\varphi$.

Since the D-action on G stabilises K, the induced action on $\mathfrak{g}$ stabilises $\mathscr{L}(K)$. But D is diagonalisable, therefore so is its action on $\mathfrak{g}$. Therefore we can decompose $\mathfrak{g}$:

$$\mathfrak{g} = \mathscr{L}(K) \oplus \mathfrak{n}$$

such that $\mathfrak{n}$ is stabilised by the D-action, and, by construction, $d\varphi$ maps $\mathfrak{n}$ isomorphically onto $\mathfrak{g}' = \mathscr{L}(G')$. In particular, if $d\varphi$ provides a 1-1 correspondence between the D-fixed points of $\mathfrak{n}$ and the D-fixed points of $\mathfrak{g}'$. In particular, $\dim(\mathfrak{c}_{\mathfrak{g}'}(D)) = \dim(\mathfrak{c}_{\mathfrak{n}}(D))$.

We now apply Proposition 2.3.1.9, to get $\mathscr{L}(C_K(D)) = \mathfrak{c}_{\mathscr{L}(K)}(D)$ and $\mathscr{L}(C_{G'}(D)) = \mathfrak{c}_{\mathfrak{g}'}(D)$. Combining this with the equality calculated above, we get

$$\begin{aligned}\dim(C_{G'}(D)) = \dim \mathfrak{c}_{\mathfrak{g}'}(D) &= \dim \mathfrak{c}_{\mathfrak{g}}(D) - \dim \mathfrak{c}_{\mathscr{L}(K)}(D) \\ &= \dim C_G(D) - \dim C_K(D) = \dim \varphi(C_G(D))\end{aligned}$$

It is clear that $\varphi(C_G(D)^\circ) \subset \varphi(C_G(D))$, and this combined with the equality of dimensions proves the result. □

2.3.2 A Useful Sequence

If G is a connected, solvable algebraic group, then Theorem 2.2.5.1 tells us that we can regard G as a closed subgroup of $T(n, K)$ for some n. As for the group $T(n, K)$, it is easy to see that there is an exact sequence

$$1 \longrightarrow U(n,K) \longrightarrow T(n,K) \xrightarrow{\pi} D(n,K) \longrightarrow 1 \qquad (2.23)$$

where π is the natural projection. We wish to use a splitting lemma here, although the Splitting Lemma we introduced as Lemma 2.1.5.1 will not hold, since these are non-commutative, linear algebraic groups we want to consider. Instead we need a modified, non-commutative version:

Lemma 2.3.2.1. *Given a short exact sequence of non-commutative linear algebraic groups*

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 1$$

if there exists algebraic group homomorphism $s : C \to B$ such that $j \circ s = \mathrm{id}_C$, then $B = A \rtimes C$ as algebraic groups. In this case, we call the sequence split.

Proof. The proof is very similar to that of Lemma 2.1.5.1 First suppose we have $b \in \ker j \cap \mathrm{im} s$, $b = s(c)$, say, for some $c \in C$. Then $c = js(c) = j(b) = 1$, and so $b = 1$, thus $\ker j \cap \mathrm{im} s$ is trivial. On the other hand, if $b \in B$, then $b = \big(b(sj(b)^{-1})\big)\big(sj(b)\big)$, and so $B \subset \ker j \cdot \mathrm{im} s$. Therefore, $B = \ker j \rtimes \mathrm{im} s = \mathrm{im} i \rtimes \mathrm{im} s$. Define a map

$$\begin{aligned} \beta : A \rtimes C &\longrightarrow B \\ (a, c) &\longmapsto i(a) \cdot s(c) \end{aligned}$$

This is certainly a homomorphism of algebraic groups. We need to show this is an isomorphism. Surjectivity is immediate from the fact that $B \subset \mathrm{im} i \cdot \mathrm{im} s$. For injectivity, suppose $\beta(a, c) = 0$, that is, $i(a) = s(c)^{-1} = s(c^{-1})$. Then $i(a) \in \mathrm{im} i \cap \mathrm{im} s$, but we saw above that this intersection is trivial, and so $i(a) = s(c) = 1$. But since both i and s are injective (the latter since $js = \mathrm{id}$), this implies $a = c = 1$, as required. □

Indeed, the sequence (2.23) is split, since certainly the inclusion map $i : D(n, K) \to T(n, K)$ is such that the composition $\pi \circ i = \mathrm{id}_{D(n,K)}$. Now observe that we can restrict the sequence (2.23) to G by virtue of the fact that G is a subgroup of $T(n, K)$ as mentioned above. Then we get our useful exact sequence:

$$1 \longrightarrow G_u \xrightarrow{\eta} G \xrightarrow{\pi} T' \longrightarrow 1 \qquad (2.24)$$

where $G_u = G \cap U(n, K)$ and $T' = \pi(G)$.

Lemma 2.3.2.2. *Consider the exact sequence (2.24):*

$$1 \longrightarrow G_u \xrightarrow{\eta} G \xrightarrow{\pi} T' \longrightarrow 1$$

If G contains a torus T with $\dim T = \dim T'$, then the sequence is split.

Proof. We claim firstly that $\pi|_T$ is a group isomorphism. Firstly we show injectivity. Certainly $T \cap G_u = \{e\}$, since T contains only semisimple elements. This tells us that $\ker(\pi) \cap T = \mathrm{im}(\eta) \cap T = \{e\}$, and so $\pi|_T$ is injective.

Secondly, it is clear that $\pi|_T$ is a morphism of algebraic groups, and so

$$\dim T = \dim \ker(\pi|_T) + \dim \mathrm{im}(\pi|_T) = \dim \mathrm{im}(\pi|_T) = \dim T'$$

Now $\mathrm{im}(\pi|_T) \subset T'$, and both these groups are connected, with the same dimension, which means they must be equal. It follows that $\pi|_T$ is surjective.

These two facts together give us that $\pi|_T$ is a group isomorphism, and so if we define $(\pi|_T)^{-1} = s$, then certainly $\pi \circ s = \mathrm{id}_{T'}$. Applying Lemma 2.3.2.1 now shows that the sequence is split.

□

2.3.3 The Unipotent Subgroup G_u

Proposition 2.3.3.1. *If G is a connected, solvable algebraic group, then G_u is a closed, connected, nilpotent normal subgroup of G, and G/G_u is a torus.*

Proof. We refer to the sequence (2.24). According to this sequence, G_u is the kernal of the morphism π, and so is closed and normal by Proposition 1.3.2.1 (1). Moreover, since $G_u = G \cap U(n, K)$, and we already know that $U(n, K)$ is nilpotent, this gives us that G_u is nilpotent (Lemma 2.2.3.5(1)). Since the sequence (2.24) is exact, we have $G/G_u = T'$. But $T' = \pi(G)$ is a closed, connected subgroup of $D(n, K)$, and hence, by Theorem 2.1.5.5, is a torus. It only remains to show that G_u is connected.

Define a group $G' = G/(G, G)$. Note that G' is commutative, and so by Theorem 2.1.1.4 can be described by the direct product $G' = G'_s \times G'_u$. Let $\varphi : G \to G'$ be the canonical quotient morphism.

Now $G_u \subset \varphi^{-1}(G'_u)$ because, as is true for any morphism, we have $\varphi(x)_u = \varphi(x_u)$. Suppose, for the reverse inclusion, that $x \in \varphi^{-1}(G'_u)$. Then $\varphi(x) = \varphi(x)_u = \varphi(x_u)$. In particular, $xx_u^{-1} \in \ker\varphi = (G, G) \subset G_u$. But $x = x_s x_u$, so $xx_u^{-1} = x_s \in G_u$ means that $xx_u^{-1} = e$, or $x \in G_u$.

Hence $\varphi^{-1}(G'_u) = G_u$. So, since (G, G) and G'_u are both connected (commutators always are, and G'_u is the unipotent part of a commutative group), it follows that G_u is connected. □

Proposition 2.3.3.2. *G_u has a chain of connected subgroups, each normal in G and each codimension one in the preceeding.*

For the proof of Proposition 2.3.3.2, we need the following lemma:

Lemma 2.3.3.3. *The group $U(n, K)$ has a chain of closed, connected subgroups, each normal in $U(n, K)$ and of codimension 1 in the preceeding one.*

Proof. Write $G = U(n, K)$. Consider pairs (i, j), where $1 \leq i, j \leq n$, and define an order on the pairs as follows:

$$(i, j) < (k, l)$$

Now for each pair (i, j), define the set

$$G_{ij} = \{x \in G \mid x_{kl} = 0, \, \forall (k,l) \leq (i,j)\}$$

It is simple to showthese are subgroups of G, and are closed and connected since they are defined by irreducible polynomial conditions. We first show that, given pairs $(i,j),(r,s)$ such that $(i,j) < (r,s)$ then $G_{ij} \supset G_{rs}$. Let $x \in U_{rs}$, so $x_{kl} = 0$ for all $(k,l) \leq (r,s)$. Suppose now that $(k,l) \leq (i,j)$. Then $(k,l) < (r,s)$, so $x_{kl} = 0$. Therefore, $x \in G_{ij}$, which in turn gives $G_{rs} \subset G_{ij}$. Note, moreover, that since this inclusion is clearly proper, $\dim G_{rs} < \dim G_{ij}$. Note also that $G_{ij} \subsetneq G$ for pairs (i,j), and so $\dim G_{ij} < \dim G = n(n-1)/2$ There are precisely $n(n-1)/2$ distinct subgroups G_{ij}. It follows that each of the dimensions of the G_{ij}'s must cycle through all the integers $0 \leq m < n(n-1)/2$, and therefore each G_{ij} is codimension 1 in the proceeding subgroup in the chain.

It remains to show that G_{ij} is normal in G, so let $g \in G$ and $x \in G_{ij}$ and let $(k,l) \leq (i,j)$. Now

$$(gxg^{-1})_{kl} = \sum_{m=1}^{n} (gx)_{km} g_{ml}^{-1} = \sum_{m=k}^{l} (gx)_{km} g_{ml}^{-1}$$

where the second equality holds because gx and g^{-1} are upper triangular. Now

$$(gx)_{km} = \sum_{p=1}^{n} g_{kp} x_{pm} = \sum_{p=k}^{m} g_{kp} x_{pm}$$

where the second equality holds because g and x are upper triangular. Together, these equations give

$$(gxg^{-1})_{kl} = \sum_{m=k}^{l} \sum_{p=k}^{m} g_{kp} x_{pm} g_{ml}^{-1}$$

But if $m < l$, then $(p,m) < (k,l) \leq (i,j)$, so $x_{pm} = 0$. Thus

$$(gxg^{-1})_{kl} = \sum_{p=k}^{l} g_{kp} x_{pl} g_{ll}^{-1} = \sum_{p=k}^{l} g_{kp} x_{pl}$$

But, if $p > k$, then $(p,l) < (k,l) \leq (i,j)$, so $x_{pl} = 0$. Thus

$$(gxg^{-1})_{kl} = g_{kk} x_{kl} = x_{kl} = 0$$

where the final equality holds since $x \in G_{ij}$. Therefore $gxg^{-1} \in G_{ij}$, which in turn tells us that G_{ij} is a normal subgroup of G. $\square$

Proof of Proposition 2.3.3.2. We refer again to the sequence (2.24), and the fact that $G_u = G \cap U(n, K)$. By Lemma 2.3.3.3, we can take a chain of normal subgroups of U_{ij}, each of codimension 1 in the one preceeding. Define groups $G_{ij} = (G \cap U_{ij})^{\circ}$ It follows that these groups are connected, normal in G, and after excluding possible repetitions, form a chain where each group is of codimension 1 in the one preceeding. $\square$

2.3.4 The Structure

Definition 2.3.4.1. Let G be an arbitrary algebraic group. Define a subgroup

$$G^\infty = \bigcap_{i \geq 0} \mathscr{C}^i G$$

Proposition 2.3.4.2. *Let G be a connected, solvable algebraic group. Then G is nilpotent if and only if G_s is a subgroup, in which case G_s is closed and connected, and $G = G_u \times G_s$.*

Proof. First we suppose that G_s is a subgroup, and show G is nilpotent. In the sequence (2.24), exactness tells us that G_s must inject into T' under π, and so $\bar{G}_s$, being a closed subgroup which is isomorphic to a subgroup of $D(n, K)$, is diagonalisable. But since $G_u = \ker \pi$, we have $\pi(G_s) = T'$, and therefore $G_s = \pi^{-1}(T')$ and so G_s is closed and connected, and hence a torus. We now apply Lemma 2.3.2.2 and Lemma 2.3.2.1 to get $G = G_s \ltimes G_u$. But G_s is normal in G, so the product is direct, that is, $G = G_s \times G_u$. Suppose $s \in G_s$. Then, for any $g = g_s g_u \in G$, we have

$$s(g_s g_u) = s(g_u g_s) = (s g_u) g_s = (g_u s) g_s = (g_u g_s) s$$

and so $G_s \subset Z(G)$. This means that $G/Z(G)$ is isomorphic to a subgroup of G_u, and so is unipotent. Corollary 2.2.4.3 then tells us that $G/Z(G)$ is nilpotent, and Lemma 2.2.3.5(3) then guarantees that G is nilpotent, as required.

For the reverse direction, suppose G is nilpotent. We need to show that G_s is a subgroup of G. Certainly, if the elements of G_s commute, then we can apply Proposition 2.1.1.2 which guarantees that G_s is simultaneously diagonalisable. This gives us, in particular, that for any two elements $x, y \in G_s$, the element xy is diagonalisable, and hence semisimple, thus showing G_s is closed under multiplication. Moreover, given any pair $x, y \in G_s$, if we can show that y commutes with all unipotent elements of G, then this guarantees that $xy = yx$. This is because $xyx^{-1}y^{-1} = z \in (G, G) \subset G_u$. Moreover, $xyx^{-1} = zy$, and if y commutes with all elements of G_u, then the right hand side of this equation will be the Jordan decomposition of the semisimple element xyx^{-1}, and therefore $e = z = xyx^{-1}y^{-1}$, as required. So we need to show that $y \in G_s$ lies in $Z(G_u)$. We use Theorem 2.3.1.1, with $U = G_u$, and $s = y$, $M = \gamma_y(G_u)$. Now the theorem says that $\gamma_y|_M$ is a bijection of M onto itself. This in turn tells us that $M \subset G^\infty = \{e\}$, the equality holding since G is assumed nilpotent. Now $\{e\} = \gamma_y(G_u)$, which says precisely $y \in Z(G_u)$, as required. □

There is an immediate corollary:

Corollary 2.3.4.3. *A connected nilpotent group has a* unique maximal torus, *namely G_s.* □

We now move on to the general case of G connected and solvable.

Theorem 2.3.4.4. *Let G be a connected solvable algebraic group. Then*

1. *G_u is a closed, connected, normal subgroup of G which includes (G, G). Furthermore, G_u has a chain of closed, connected subgroups each normal in G and each having codimension one in the next.*

2. *The maximal tori of G are conjugate.*

3. *If T is a maximal torus, then $G = T \ltimes G_u$.*

For the proof of Theorem 2.3.4.4, we use the following lemma:

Lemma 2.3.4.5. *Let G be a connected, solvable algebraic group. Let T be a maximal torus of G, with $s \in G$ semisimple. Then there exists an element $x \in G^{\infty}$ such that $xsx^{-1} \in T$.*

Proof. We prove the lemma by induction on $\dim G$. Certainly, if $\dim G = 0$, then the result is trivially true, so suppose it is true for any connected, solvable algebraic groups of dimension less than n. If G is nilpotent, then Corollary 2.3.4.3 tells us that $T = G_s$, and therefore $s \in T$ by definition.

Suppose now that G is not nilpotent, which implies immediately that G^{∞} is nontrivial. Indeed, Proposition 2.2.1.3 (1) tells us that G^{∞} is connected, and, since $(G, G) \subset G_u$ by Proposition 2.3.3.1, and since certainly $G^{\infty} \subset (G, G)$, it follows that $G^{\infty} \subset G_u$, and so is unipotent, and thus, in turn, is nilpotent.

Now, Lemma 2.2.3.5 (3) tells us that $N = Z(G^{\infty})$ is nontrivial, and equal to $\mathscr{C}^m G^{\infty}$ for some m. Now, the fact that G^{∞} is connected, combined with Proposition 2.2.1.3(1), tells us that $\mathscr{C}^m(G^{\infty}) = N$ is connected, and so, being nontrivial, has positive dimension.

We aim now to show that N is normal in G. Supposing that $z \in N, y \in G$ and $x \in G^{\infty}$, we wish to show that $yzy^{-1} \in N$, or that yzy^{-1} commutes with x. Now certainly G^{∞} is normal in G, since it is the intersection of normal subgroups of G. Therefore,

$$x(yzy^{-1}) = y(y^{-1}xy)zy^{-1} = yz(y^{-1}xy)y^{-1} = (yzy^{-1})x$$

where the middle equality is since $(y^{-1}xy) \in G^{\infty}$, and so commutes with z. This shows that N is normal in G.

Consider now the quotient homomorphism

$$\begin{aligned} \eta : G &\longrightarrow G/N = G' \\ g &\longmapsto gN \end{aligned}$$

We claim that $\eta(G^{\infty}) = G'^{\infty}$. In the first place, if $x \in G^{\infty}$, then $x \in \mathscr{C}^i(G)$ for all i, but $\eta(\mathscr{C}^i(G)) = \mathscr{C}^i(\eta(G))$, so $\eta(x) \in \mathscr{C}^i(G')$ for all i. Thus $\eta(G^{\infty}) \subset G'^{\infty}$. For the reverse inclusion, suppose $xN \in G'^{\infty}$. This means, for each i, $xN \in \mathscr{C}^i(\eta(G)) = \eta\mathscr{C}^i(G)$, so $xN = yN$ for some $y \in \mathscr{C}^i(G)$. But then $xy^{-1} \in N \subset G^{\infty}$, and, in particular, $xy^{-1} \in \mathscr{C}^i(G)$. But $\mathscr{C}^i(G)$ is a group, so $x = (xy^{-1})y \in \mathscr{C}^i(G)$. But since i was arbitrary, it follows that $x \in G^{\infty}$, and so $G'^{\infty} \subset \eta(G^{\infty})$, as required.

Let $S = \eta(T)$. Then S is a maximal torus of G', since $\eta^{-1}(S)$ is a torus of G which contains T, and so is equal to T. Now G' is solvable, being a homomorphic image of a solvable group, and has dimension less than G, since N is nontrivial. By our induction hypothesis, then, $\eta(x)\eta(s)\eta(x)^{-1} \in S$, for some $x \in G^\infty$, and so $xsx^{-1} \in NT$. Write $s' = xsx^{-1} = nt$, for some $n \in N, t \in T$. Certainly s' is semisimple, being conjugate to s.

Now, we saw above that N is unipotent, and certainly $t \in T$ normalises N, since N is normal in G. So the conditions of Theorem 2.3.1.1 are satisfied, and in particular, applying Theorem 2.3.1.1(2), we have $n = z\gamma_t(u)$ for some $z \in C_N(t)$ and some $u \in N$. Therefore

$$s' = nt = (zutu^{-t}t^{-1})t = z(utu^{-1}) \tag{2.25}$$

Now $zu = uz$ since $z, u \in N = Z(G^\infty)$, and $zt = tz$ since $z \in C_N(t)$. It follows that $z(utu^{-1}) = (utu^{-1})z$, that is, the bracketed elements of the far right hand side of (2.25) commute. Furthermore, z is unipotent since it lies in N, and utu^{-1} is semisimple, since it is conjugate to t. Therefore the bracketed far right hand side of (2.25) is the Jordan decomposition of the far left. But s' is semisimple, which means $z = e$ and $s' = utu^{-1}$. But this means $s = (x^{-1}u)t(x^{-1}u)^{-1}$, and so s is conjugate to an element of T under G^∞, as required. □

Proof of Theorem 2.3.4.4. (1) was proven in Propositions 2.3.3.1 and 2.3.3.2. Consider (2). Let S be another maximal torus. Then Proposition 2.3.1.8 says there is an element $s \in S$ such that $C_G(s) = C_G(S)$. By Lemma 2.3.4.5, there exists an element $x \in G^\infty$ such that $s' = xsx^{-1} \in T$. Defining $S' = xSx^{-1}$, it is clear S' is a maximal torus of G, and, moreover, $G_G(s') = C_G(S')$. Indeed, since $s' \in T$, we have $T' \subset C_G(s') = C_G(S')$. Suppose now that $\tilde{s} \in S', \tilde{t} \in T$. Then $\{\tilde{s}, \tilde{t}\}$ is a commutative set consisting of semisimple elements, and so Proposition 2.1.1.2 tells us that this set is diagonalisable, which in turn implies that $\tilde{s}\tilde{t}$ is semisimple. Therefore $(ST)^\circ$ is a connected group which is commutative and consists of semisimple elements, and so is a torus. But clearly $S', T \subset (S'T)^\circ$, and since S', T are maximal, it follows that $S' = T = (S'T)^\circ$. Therefore $xSx^{-1} = T$, but S was an arbitrary maximal torus, so we can conclude that all maximal tori are conjugate.

Consider (3). If G is nilpotent, then the result is already proven in Proposition 2.3.4.2, so suppose G is not nilpotent. We see that by Lemma 2.3.2.2, all we need to do is to find a torus in G of dimension equal to that of T'. We show this by induction on $\dim G$. Certainly, if $\dim G = 0$, then the result is trivially true, so suppose $\dim G = n$, and that the result holds for all solvable connected groups of dimension less than n. If, for all $s \in G_s$, we have $C_G(s) = G$, then G_s is a commutative set which consists of semisimple elements, and by Proposition 2.1.1.2 it is diagonalisable, which in turn means that it is a subgroup. But then Proposition 2.3.4.2 tells us that G is nilpotent, which is contrary to our assumption. Therefore there is a semisimple element s such that $C_G(s) \subsetneq G$, and in particular, $\dim C_G(s)^\circ < \dim G$. Write $H = C_G(s)^\circ$.

If we set $D = \overline{<s>}$, then D is a d-group. Indeed, by Lemma 2.3.4.5, $s \in xTx^{-1}$ for some $x \in G$, and so s is contained inside a torus. But tori are closed, and so $D \subset xTx^{-1}$, which implies D is commutative and consists of semisimple elements. Next, we let $h \in H$, and define a little morphism as follows:

$$\begin{aligned}\eta : G &\longrightarrow G \\ g &\longrightarrow ghg^{-1}\end{aligned}$$

Now $\varphi^{-1}(h)$ is certainly closed, and, moreover, $< s > \subset \varphi^{-1}(h)$. It follows, therefore, that $D \subset \varphi^{-1}(h)$, which tells us that D acts trivially on H, or $H \subset G^D$. On the other hand, $s \in D$, so $G^D \subset C_G(s)$. We can conclude that $(G^D)^\circ = H$.

Now consider the quotient morphism

$$\pi : G \longrightarrow G/G_u = T'$$

Moreover, since T' is commutative, $\pi(D)$ acts trivially on T', and so π is equivariant with respect to the action of D. Applying Corollary 2.3.1.11 tells us that

$$\pi(H) = \pi\left((G^D)^\circ\right) = \pi\left((T'^D)^\circ\right) = T'$$

By induction, then, there is a torus $T \subset H$ such that $\dim T = \dim T'$. But $S \subset G$, and so the preconditions of Lemma 2.3.2.2 are met, therefore $G = T \ltimes G_u$ as required.

□

Corollary 2.3.4.6. *Let G be a connected solvable group. Then each semisimple element (respectively, unipotent) of G lies in a maximal torus (respectively, in a maximal connected unipotent subgroup).*

Proof. These statements we can be deduced from Lemma 2.3.4.5 and Theorem 2.3.4.4 (1). Indeed, if $u \in G$ is unipotent, then u lies in G_u, which is a maximal connected unipotent subgroup. On the other hand, if T is a maximal torus of G and $s \in G$ is semisimple, then Lemma 2.3.4.5 says there is an element $x \in G$ such that $xsx^{-1} \in T$, and so s lies in xTx^{-1}, which is certainly a torus, since it is connected, commutative, and consists of semisimple elements. It follows that s lies in a maximal torus. □

We note here that Corollary 2.3.4.6 has a more general formulation, given later as Theorem 3.1.6.2, also called the Density Theorem, in which the hypothesis can be softened to allow any connected algebraic group, solvable or otherwise.

Proposition 2.3.4.7. *Let H be a (not necessarily closed) subgroup of a connected solvable algebraic group G, with H consisting of semisimple elements. Then*

1. *H is included in some maximal torus; in particular, any subtorus of G lies in a maximal torus.*

2. *$C_G(H) = N_G(H)$ is connected.*

Proof. Let $\pi : G \to G/G_u$ be the quotient morphism. Certainly $\ker \pi = G_u$, which means $H \cap \ker \pi = \{e\}$, since H consists of semisimple elements, and this in turn means $\pi|_H$ is injective. Moreover, applying Theorem 2.3.4.4 (2) to the group G/G_u tells us that this group must be a torus, and in particular is commutative. In turn, $\pi(H)$ is commutative, and therefore H itself is commutative by the injectivity of $\pi|_H$.

Suppose now that $H \subset Z(G)$. Then $C_G(H) = G = N_G(H)$ is connected. Suppose now that $h \in H$. Then, by Corollary 2.3.4.6, there is a maximal torus T of G such that $h \in T$. Moreover, if $h' \in H$ is another element, then $h' \in T'$ where T' is another maximal torus of G. But, by Theorem 2.3.4.4 (2), there is an element $x \in G$ such that $T' = xTx^{-1}$. Then $h' = xtx^{-1}$ for some $t \in T$. But $H \subset Z(G)$, so $h' = t \in T$, which means $H \subset T$, and the result is proven.

Suppose now that H is not contained in $Z(G)$, so let $h \in H$ with $h \notin Z(G)$. Applying Corollary 2.3.4.6 again, let T be a maximal torus of G which contains the semisimple element h. We wish to prove two statements at this point, using induction. Firstly, we claim that H is conjugate to a subgroup of T under an element taken from $C_G(s)$, and secondly that $C_G(H)$ is closed. Now, certainly these results are trivial in the case $\dim G = 0$, since G is assumed to be connected, and so $G = H = \{e\}$. Suppose now that these results hold for solvable groups of dimension less than $\dim G = n$.

Now, since T is commutative and contains h, we have $T \subset C_G(h)$, and in particular, T is a maximal torus of $C_G(h)$. Moreover, we showed above that H itself is commutative, and since $h \in H$, it follows that $H \subset C_G(h)$ Now $C_G(h)$ is certainly solvable, being a subgroup of the solvable group G, so, applying Theorem 2.3.4.4(2) again tells us that $C_G(h) = T \ltimes [C_G(h)]_u$. But $[C_G(h)]_u = C_G(h) \cap G_u = C_{G_u}(h)$. We can apply Theorem 2.3.1.1 to this last group, to show that it is connected. It follows that $C_G(h)$ is connected, being the semidirect product of two connected groups. Note now that, since $h \in C_G(h)$ yet $h \notin Z(G) \subset G$, we have $C_G(h) \subsetneq G$, and so, both groups being connected, it follows $\dim C_G(h) < \dim G$. By our induction hypothesis, then, H is conjugate under $C_{C_G(h)}(H)$ to a subgroup S of T, and $C_{C_G(h)}(H)$ is connected. But looking at the groups closely, it is apparenty that $C_{C_G(h)}(H) = C_G(H)$, so H is conjugate under $C_G(H)$ to a subgroup S of T, and $C_G(H)$ is connected.

Finally we need to show that $C_G(H) = N_G(H)$. Certainly $C_G(H) \subset N_G(H)$, so for the reverse inclusion, we return to the quotient morphism π from the first part of this proof. Let $x \in N_G(H)$ and $y \in H$. Then

$$\pi(xyx^{-1}) = \pi(x)\pi(y)\pi(x)^{-1} = \pi(y)$$

where the latter equality is by commutativity of $\pi(H)$. By injectivity of $\pi|_H$, we have $xyx^{-1} = y$, which implies $x \in C_G(H)$, which in turn gives $C_G(H) = N_G(H)$, as required. $\square$

Chapter 3

Reductive Groups

We will see in Chapter 5 that the Classical Groups are examples of *semisimple* groups. We will learn what this means in this chapter, and in fact introduce the more general notion of a *reductive* group. Reductive groups have a particularly elegant structure theory, which we present in all its glory in §3.4. Ultimately, reductive groups are more convenient to work with than semisimple groups, since reductive groups beget reductive subgroups quite readily, as we will see in §3.4.2, especially. Hence reductive groups admit elegant inductive style arguments, which are really the key to revealing their structure.

The chapter closes with a brief description of the Bruhat decomposition of semisimple groups, as yet another way of understanding the structure of such a group.

3.1 Borel Subroups

Since we now know a great deal about solvable groups from Chapter 2, it is reassuring to know that solvable groups play a large role in understanding the structure theory of the groups we are ultimately interested, namely the reductive groups. This section focuses on the properties of the maximal solvable groups, the Borel subgroups, upon which the entire structure theory is in a sense hung.

3.1.1 Radicals

Definition 3.1.1.1. Let G be an arbitrary algebraic group. The largest connected, normal, solvable subgroup of G is the *radical* of G, denoted by $R(G)$.

Note that, by Lemma 2.2.5.2, $R(G)$ is closed. Before going any further, we should ensure that such a subgroup exists, and is well defined. Indeed, given any two closed, connected, normal, solvable subgroups $A, B \subset G$, Corollary 1.3.1.10 says that AB is closed, it is self evidently normal, and Lemma 2.2.2.5(3) says that AB is solvable. Since $A, B \subset AB$, it is clear that we can find a unique radical by simply taking the product of all connected, normal, solvable subgroups.

Example 20. For $G = GL(n, K)$, we will see in Section 5.1.3 that $R(G) \cong \mathbb{G}_m$.

Definition 3.1.1.2. If $R(G)$ is trivial, and $G \neq \{e\}$ is connected, we call G *semisimple.*

A *semisimple Lie algebra* is one which has trivial maximal solvable ideal (see, for example, [5]), hence is analagous to the above definition for semisimple algebraic groups.

Definition 3.1.1.3. The subgroup of $R(G)$ consisting of all its unipotent elements is called the *unipotent radical* of G, and is denoted by $R_u(G)$.

Example 21. Following on from Example 20, we see that for $G = GL(n, K)$, $R_u(G) = \{e\}$.

At this point, we will introduce a new concept, to help us deal with radicals:

Definition 3.1.1.4. Let H be a subgroup an algebraic group G. We say that H is a *characteristic subgroup* if it is invariant under any isomorphism of algebraic groups $\varphi : G \to G$.

The examples of most interest to us are the following:

Lemma 3.1.1.5. *(a) Let G be an algebraic group. Then $R(G)$ is a characteristic subgroup of G.*

(b) Let G be an algebraic group. Then $R_u(G)$ is a characteristic subgroup of $R(G)$.

Proof. (a): Let $\varphi : G \to G$ be a group automorphism. Since the homomorphic image of a connected, solvable group is itself connected and solvable, we only need to show that $\varphi(R(G))$ is normal in G. Suppose $g \in R(G)$ and $y \in G$. Then $y = \varphi(x)$ for some $x \in G$ since φ is an automorphism, and therefore

$$y\varphi(g)y^{-1} = \varphi(x)\varphi(g)\varphi(x^{-1}) = \varphi(xgx^{-1}) = \varphi(g')$$

where $xgx^{-1} = g' \in R(G)$ since $R(G)$ is normal in G. Therefore $\varphi(R(G))$ is normal in G. We have therefore shown that $\varphi(R(G))$ is a connected, solvable, normal subgroup of G, and so is contained in the maximal such subgroup, which is $R(G)$ by definition. Therefore φ stabilises $R(G)$, and so the latter is a characteristic subgroup.

(b): Let $\varphi : R(G) \to R(G)$ be an isomorphism of algebraic groups. Then

$$\varphi(R_u(G)) = \varphi(R(G)_u) = \varphi(R(G))_u = R(G)_u$$

where the last equality is since φ is surjective. Therefore $R_u(G)$ is a characteristic subgroup.

$\square$

Proposition 3.1.1.6. *If H is a characteristic subgroup of an algebraic group G, then it is normal in G. Moreover, if G itself is normal in some ambient algebraic group G', then H is normal in G'.*

Proof. Certainly $\mathrm{Int}(x)$ is an automorphism of algebraic groups on G for any $x \in G$. Since H is characteristic, we have $xHx^{-1} = H$, and so H is normal in G. Suppose $y \in G'$. Then $\mathrm{Int} y$ maps G to G, since G is normal in G'. Therefore, $\mathrm{Int}(y)$ is an automorphism of G, and so stabilises H. That is, $yHy^{-1} = H$, and so H is normal in G'. □

Proposition 3.1.1.7. *$R_u(G)$ is the largest normal, connected, unipotent subgroup of G.*

Proof. $R_u(G)$ is connected and normal in $R(G)$, by Theorem 2.3.4.4. Since $R_u(G)$ is a characteristic subgroup of $R(G)$ (Lemma 3.1.1.5(b)), we can use Proposition 3.1.1.6 to tell us that $R_u(G)$ is in fact normal in G.

Suppose now that H is a normal, connected, unipotent subgroup of G such that $R_u(G) \subset H$. Then $R(G) \cdot H$ is normal and solvable, by Lemma 2.2.2.5(3), and is connected, which implies $R(G) \cdot H \subset R(G)$, whence $H \subset R(G)$, and therefore $H \subset R_u(G)$, as required. □

Definition 3.1.1.8. Let G be an algebraic group. If $R_u(G)$ is trivial, and $G \neq \{e\}$ is connected, then we say that G is *reductive.*

Example 22. A torus is reductive.

Example 23. A semisimple group is (by definition) reductive.

Lemma 3.1.1.9. *Let G be connected and reductive. Then $R(G) = [Z(G)]^\circ$, is a torus, and has finite intersection with (G, G).*

Proof. Since $R(G)$ is connected and solvable, Theorem 2.3.4.4 tells us that $R(G) = T \ltimes R_u(G)$. But since $R_u(G) = \{e\}$, this implies $R(G)$ is a torus, which we write $R(G) = S$. Now, by Corollary 2.1.6.3, and the fact that $S = R(G)$ is normal in G, we have

$$G = G^\circ = (N_G(S))^\circ = (C_G(S))^\circ$$

since S is a torus. This then gives us $S \subset Z(G)^\circ$. For the reverse inclusion, note firstly that, since the largest connected unipotent group $R_u(G)$ is trivial, then certainly $Z(G)_u^\circ = \{e\}$. But we can apply Theorem 2.1.1.4, since $Z(G)$ is commutative. Hence $Z(G)^\circ = Z(G)_u^\circ \times Z(G)_s^\circ = Z(G)_s^\circ$. Therefore $Z(G)^\circ$ is a connected solvable subgroup, and hence $Z(G)^\circ \subset R(G) = S$, as required.

For the finiteness assertion, we use Theorem 1.4.4.8 to embed G in some $GL(V)$, and we can then emulate the argument given in the proof of Proposition 2.3.1.8. In particular, since $S = R(G)$ is a torus, we showed in the proof of Proposition 2.3.1.8 that

$$C_{GL(V)}(S) = \prod GL(V_\alpha)$$

where α ranges over the weights of S in V, that is, the characters $\alpha \in X(S)$ such that the space V_α is nonzero. Now, certainly $G \subset C_{GL(V)}(S) = \prod GL(V_\alpha)$, since $S \subset Z(G)$. It follows that $(G, G) \subset \prod SL(V_\alpha)$, since commutators must have determinant one.

Additionally, since $S \subset G$, we have $S \subset \prod GL(V_\alpha)$. But by construction, elements of S act on each V_α by scalar multiplication, namely $s.v = \alpha(s)v$ for all $s \in S$ and $v \in V_\alpha$, and so elements of S are block diagonal matrices, each of whose blocks is a scalar matrix. Elements of $S \cap (G, G)$ are therefore block diagonal matrices, each of whose blocks is a scalar matrix of determinant one. There are only finitely many scalar matrices of determinant one of any given size, namely those with roots of unity as their entries, and so $S \cap (G, G)$ is contained within a finite set, and so is itself finite, as required. □

Corollary 3.1.1.10. *Let G be connected and reductive. Then (G, G) is semisimple.*

Proof. By Lemmas 3.1.1.5(a) and 3.1.1.9, $R(G, G) = Z(G, G)^\circ$ is a characteristic subgroup of (G, G). But it is easy to show that (G, G) is also a characteristic subgroup of G, and so $R(G, G)$ is a characteristic subgroup of G, and is therefore normal in G. Certainly $R(G, G)$ is connected and solvable, so $R(G, G) \subset R(G)$ by maximality of the radical of G. Now, since and $R(G) = Z(G)^\circ$ by another application of Lemma 3.1.1.9, and certainly $R(G, G) \subset Z(G)$, we have

$$R(G, G) \subset Z(G) \cap (G, G)$$

But this larger set is finite, by the second part of Lemma 3.1.1.9. Since $R(G, G)$ is finite and connected, it must be trivial, as required. □

Corollary 3.1.1.11. *Let H be a closed, normal subgroup of an algebraic group G. Then H is semisimple if G is, and is reductive if G is.*

Proof. By Lemma 3.1.1.5(a), $R(H)$ is a characteristic subgroup of H, and this combined with Proposition 3.1.1.6 tells us that $R(H)$ is normal in G. Since $R(H)$ is connected and solvable, it follows that $R(H) \subset R(G)$. Therefore, if G is semisimple, then $R(H) \subset R(G) = \{e\}$, while if G is reductive, then $R_u(H) \subset R_u(G) = \{e\}$, and so the result is proven. □

Example 24. In §5.1.3 of Chapter 5, we will show that the radical of $G = GL(n, K)$ is the group which consists of scalar matrices, and therefore G is reductive. Moreover, $SL(n, K)$ is normal in G, and so Corollary 3.1.1.11 gives us that $SL(n, K)$ is reductive. We will also calculate the centre of $SL(n, K)$ (see §5.2.1), and will see that it is in fact finite, which therefore means that $SL(n, K)$ is semisimple. We will explore these and other examples in much greater detail in Chapter 5.

3.1.2 Complete Varieties

Definition 3.1.2.1. A variety X is *complete* if, for all varieties Y, the projection map

$$\mathrm{pr}_2 : X \times Y \longrightarrow Y$$

is closed, that is, sends closed sets to closed sets.

Completeness is often thought of as as an algebraic geometric analogy for the topological notion of compactness. See, for example, page 45 of [6]. We now describe some important properties of complete varieties.

Example 25. If X consists of a single point x, then X is complete. Indeed, suppose Y is a variety, and let $F \subset X \times Y$ be closed subset. Then $F = \{x\} \times F'$ where F' is closed in Y. Then $\mathrm{pr}_2(F) = F'$, and so is closed in Y, as required.

Lemma 3.1.2.2. *A variety X is complete if and only if all of its irreducible components are.*

Proof. Suppose $X = X_1 \cup \cdots \cup X_m$ with X_i the irreducible components of X. Suppose further that each X_i is complete, and let Y be an arbitrary variety, and $\mathrm{pr}_2 : X \times Y \to Y$ be the projection morphism. If $F \subset X \times Y$ is closed, then $F_i = F \cap (X_i \cap Y)$ is closed in $X_i \times Y$ for each i. Now,

$$\mathrm{pr}_2(F) = \mathrm{pr}_2(\bigcup F_i) = \bigcup \mathrm{pr}_2(F_i)$$

$\mathrm{pr}_2(F)$ is thus closed in Y, being the finite union of closed sets, and so X is complete.

Suppose now that X is complete, and let F_i be a closed subset of $X_i \times Y$, for arbitrary Y. Then, since $X_i \times Y$ is closed in $X \times Y$, we have that F_i is closed in $X \times Y$. Therefore, since X is complete, $\mathrm{pr}_2(F_i)$ is closed in Y, making X_i complete, as required. □

Example 26. If X is finite, then Lemma 3.1.2.2 and Example 25 together show that X is complete.

Lemma 3.1.2.3. $\mathbf{A}^1$ *is not complete.*

Proof. Consider the projection $\mathrm{pr}_2 : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$. Let $F = \{(x,y) \in K^2 \,|\, xy - 1 = 0\}$. Certainly F is a closed subset of $\mathbf{A}^2$, but $\mathrm{pr}_2(F) = \mathbf{A}^1 \backslash \{0\}$, which is an open subset of $\mathbf{A}^1$, and certainly not closed. Therefore $\mathbf{A}^1$ cannot be complete. □

Proposition 3.1.2.4.

1. *A closed subvariety of a complete variety is complete.*
2. *If X and Y are complete, then so is $X \times Y$.*
3. *If $\varphi : X \to Y$ is a morphism of varieties, with X complete, then the image is closed in Y and complete.*

4. If X is complete, and is a subvariety of Y, then X is closed in Y.

5. If X is irreducible and complete, then any regular function on X is constant.

6. A complete affine variety is finite, and therefore of dimension 0.

Proof. For (1), we note that, if $Z \subset X$ is a closed subvariety, then a closed set in $Z \times Y$, for Y some variety, will be closed in $X \times Y$, and so will project onto a closed set of Y.

For (2), suppose Z is an arbitrary variety, and note that the projection $(X \times Y) \times Z \to Z$ is simply the composition of two closed maps, namely the projection $X \times (Y \times Z) \to Y \times Z$, composed with the projection $Y \times Z \to Z$, and so, since the composition of two closed maps is clearly closed itself, the result follows.

For (3), let $\Gamma = \{(x, \varphi(x) \mid x \in X\} \subset X \times Y$ be the graph of φ. Since X and Y are both varieties, by definition Γ is closed in $X \times Y$. Since X is complete, $\mathrm{pr}_2(\Gamma) = \varphi(X)$ is closed. Moreover, it is easy to see that Γ and X are isomorphic, so Γ is complete. Now $\varphi(X) \cong \{x\} \times \varphi(X)$ is closed in Γ, and so, by (1), $\varphi(X)$ is complete.

Now (4) follows from (3), applied to the embedding $X \hookrightarrow Y$.

For (5), let X be an irreducible, complete variety, with $\varphi \in K[X]$. That is, $\varphi : X \to \mathbf{A}^1$ is a morphism. According to (3), $\varphi(X)$ is complete and closed in $\mathbf{A}^1$. By Lemma 3.1.2.3, $\varphi(X) \subsetneq \mathbf{A}^1$, which implies $\dim \varphi(X) < \dim \mathbf{A}^1 = 1$. Therefore $\varphi(X)$ is a singleton, and so φ must be a constant map.

Now (6) follows from (5). Indeed, suppose X is irreducible, affine and complete. Then each element $f \in K[X]$ is constant, and so $K[X] = K$, which guarantees that X is a singleton, as required. $\square$

Theorem 3.1.2.5. *Projective varieties are complete.*

Proof. It suffices to prove that $\mathbb{P}^n$ is complete by Proposition 3.1.2.4(1), so we need to show that $\mathrm{pr}_2 : \mathbb{P}^n \times Y \to Y$ is a closed map for any variety Y. We may also assume that Y is irreducible and affine, so let $R = K[Y]$ be its affine algebra. We will also use the notation $U_i = \mathbb{P}^n_i \times Y$, where $\mathbb{P}^n_i$ denotes the open affine subset of $\mathbb{P}^n$ which has nonzero i^{th} coefficient.

Take a closed set $Z \subset \mathbb{P}^n \times Y$. We need to show that $\mathrm{pr}_2(Z)$ is closed in Y. Equivalently, we need to show that its complement is open in Y, so take an element $y \in Y - \mathrm{pr}_2(Z)$.

Let $S = R[X_0, \ldots, X_n]$. Since S is a polynomial ring, there is a natural grading by dimension on S, which we denote by $S = \bigoplus S_m$. For each m, we define a subset I_m of S_m as follows:

$$I_m = \{g(X_0, \ldots, X_n) \in S_m \mid g(X_0/X_i, \ldots, 1, \ldots, X_n/X_i) \in \mathscr{I}(Z_i), 0 \leq i \leq n\}$$

where $Z_i = U_i \cap Z$. Now we define $I = \bigoplus I_m$. Since $g \in I_m, h \in S_p$ implies $hg \in S_{m+p}$, it follows easily that I is an ideal of S, and, moreover, being graded, it is a homogeneous ideal.

Fix i between 0 and n, and let $f(X_0/X_i, \ldots, X_n/X_i) \in \mathscr{I}(Z_i)$. We claim that $X_i^p f(X_0, \ldots, X_n) \in I_p$ for some p. Indeed, if $m = \deg\big(f(X_0/X_i, \ldots, X_n/X_i)\big)$, then $X_i^m f(X_0, \ldots, X_n) \in S_m$. Furthermore, $(X_i/X_j)^m f(X_0, \ldots, X_n) \in R_j$ vanishes on $Z_i \cap U_j = Z_j \cap U_i$. Suppose that $z \in Z_j - U_i$. Then z has i^{th} coordinate zero, and so $(X_i/X_j)^{m+1} f(X_0, \ldots, X_n)$ vanishes at z, since each term of this polynomial has X_i as a factor. It follows that $(X_i/X_j)^{m+1} f(X_0, \ldots, X_n)$ vanishes on Z_j. But j was arbitrarily chosen, so $(X_i/X_j)^{m+1} f(X_0, \ldots, X_n) \in \mathscr{I}(Z_j)$ for all j, and therefore the $X_i^{m+1} f(X_0, \ldots, X_n)$ lies in I_{m+1}.

Since $y \notin \mathrm{pr}_2(Z)$, it follows that $Z_i \cap (\mathbb{P}_i^n \times \{y\}) = \emptyset$. Moreover, Z_i and $\mathbb{P}_i^n \times \{y\}$ are both closed in U_i, and so

$$\emptyset = Z_i \cap (\mathbb{P}_i^n \times \{y\}) = \mathscr{V}\big(\mathscr{I}(Z_i)\big) \cap \mathscr{V}\big(\mathscr{I}(\mathbb{P}_i^n \times \{y\})\big) = \mathscr{V}\big(\mathscr{I}(Z_i) + \mathscr{I}(\mathbb{P}_i^n \times \{y\})\big) \tag{3.1}$$

and so, by Hilbert's Nullstellensatz, $\mathscr{I}(Z_i) + \mathscr{I}(\mathbb{P}_i^n \times \{y\}) = R_i$. Note also that $\mathscr{I}(\mathbb{P}_i^n \times \{y\} = \mathscr{I}_R(y) R_i$, and, moreover, $\mathscr{I}_R(y)$ is a maximal ideal of R, and we will denote it by M. Returning to equation (3.1), since $1 \in R_i$, we have

$$1 = f_i + m_i g_i$$

for $f_i \in \mathscr{I}(Z_i), m_i \in M$ and $g_i \in R_i$. We saw above that there exists an integer p_i such that $X_i^{p_i} f_i \in I$. We could also take p_i to be sufficiently large such that $X_i^{p_i} g_i \in S_{p_i}$. Plugging this into the above equation, we see

$$X_i^{p_i} = X_i^{p_i} f_i + m_i X_i^{p_i} g_i \in I_{p_i} + M S_{p_i}$$

If we take $p = \max\{p_i\}$, then

$$X_i^p = X_i^p f_i + m_i X_i^p g_i \in I_p + M S_p \tag{3.2}$$

Suppose now that $h \in S_m$, homogeneous of degree m. If we take m to be sufficiently large, in particular, if $m \geq (n+1)p$, then each term of h has X_i^p as a factor for some i. Plugging this in to equation (3.2) tells us that

$$h \in I_m + M S_m \subset S_m$$

And so $S_m \subset I_m + M S_m$. Now $I_m \subset S_m$ by defintion, and since $M \subset R$, we have $M S_m \subset S_m$. Therefore $I_m + M S_m \subset S_m$, and this combined with the inclusion just proven, gives

$$S_m = I_m + M S_m \tag{3.3}$$

Now I_m is an R-ideal of the R-module S_m. Therefore, S_m/I_m is an R-module. Taking this quotient, equation (3.3) becomes

$$S_m/I_m = M S_m/I_m$$

Since R is a polynomial ring, the R-module S_m/I_m is finitely generated, and so we can apply Nakayama's Lemma ([1] Corollary 2.5) which tells us that there exists $f = 1 + a \subset R$ for some $a \in M$ such that $fS_m/I_m = 0$. That is, $f \notin M$ and $fS_m \subset I_m$. In particular, $fX_i^m \in I_m$, for all i, which means

$$f(X_i/X_i)^m = f \in \mathscr{V}(Z_i)$$

Since $f \in R$, this in fact tells us that $f \in \mathscr{V}(\mathrm{pr}_2(Z_i))$ for all i, which in turn gives $f \in \mathscr{V}(\mathrm{pr}_2(Z))$. Now $Y_f = \{y \in Y \mid f(y) \neq 0\}$ is open in Y, and since $f \notin M = \mathscr{I}(y)$, we have $f(y) \neq 0$, so Y_f is a neighbourhood of y. Moreover, since $f \in \mathscr{V}(\mathrm{pr}_2(Z)$, we see that $\mathrm{pr}_2(Z) \cap Y_f = \emptyset$. But this implies that $Y - \mathrm{pr}_2(Z)$ is open in Y, as required. □

Example 27. The flag variety discussed in Example 12 of Chapter 1 is projective (see §10.3 of [2]), and hence is complete.

Lemma 3.1.2.6. *Suppose X, Y and Z are irreducible varieties, with X also complete, and $\varphi : X \times Y \to Z$ a morphism. For each $x \in X, y \in Y$, write $\varphi_y(x) = \varphi(x, y)$, so φ_y is a morphism from X into Z. If there is an element $a \in Y$ such that φ_a is constant, then φ_y is constant for each $y \in Y$.*

Proof. Let Γ be the graph of φ, that is,

$$\Gamma = \{(x, y, \varphi(x, y)) \mid x \in X, y \in Y\} \subset X \times Y \times Z$$

We know that Γ is closed in $X \times Y \times Z$ since the graph of any morphism of varieties is closed, as a consequence of the Hausdorff axiom; see, for example, Proposition 2.5 of [6], which gives this result in the slightly more general setting of a morphism $f : X_1 \to X_2$, with X_2 a variety, and X_1 a prevariety. Therefore, if we consider the projection map

$$\alpha : X \times (Y \times Z) \longrightarrow Y \times Z$$

then $\alpha(\Gamma) = A = \{(y, \varphi(x, y) \mid x \in X, y \in Y\}$ is closed in $Y \times Z$, by completeness of X. Now we consider the projection map

$$\pi : A \longrightarrow Y$$

The set $\pi^{-1}(a) = \{(a, \varphi(x, a) \mid x \in X\}$ consists of only one point, by assumption. We now apply Theorem 4.2.5 of [9], which says, given a surjective morphism $f : X_1 \to X_2$ of irreducible varieties with a finite fibre $f^{-1}(x)$ for some $x \in X_2$, then $\dim X_1 = \dim X_2$. Applied to our situation, this tells us that $\dim A = \dim Y$. Now, for each $x \in X$, we define a subset A_x as follows:

$$A_x = \{(y, \varphi(x, y) \mid y \in Y\}$$

Now A_x is the graph of some morphism $Y \to Z$, and therefore is closed in $Y \times Z$, and is isomorphic to Y. The latter fact tells us that A_x is irreducible, and that $\dim A_x = \dim A$. This is enough to guarantee that $A_x = A$, whence φ_y is constant for each $y \in Y$, as required. □

Proposition 3.1.2.7. *Let G be a complete, connected algebraic group. Then G is commutative.*

Proof. We simply apply Lemma 3.1.2.6 to the case $X = Y = Z = G$, and the $\varphi(x, y) = xyx^{-1}$. Then certainly $\varphi_e(x) = e$ for all $x \in G$, whence φ_y is constant for each y. In particular, for each $x \in G$

$$y = \varphi_y(e) = \varphi_y(x) = xyx^{-1}$$

whence $xy = yx$ for all $x, y \in G$, as required. □

Definition 3.1.2.8. Complete, connected algebraic groups are called *abelian varieties.*

We do not concern ourselves much with abelian varieties here.

3.1.3 Parabolic Subgroups

Definition 3.1.3.1. Let $P \subset G$ be a closed subgroup. P is called *parabolic in* G if G/P is complete.

Lemma 3.1.3.2. *Let P be a parabolic subgroup of G. Then G/P is projective.*

Proof. By Corollary 1.5.3.3, we know that G/P is quasi-projective, that is, an open subvariety of a projective variety. This combined with Proposition 3.1.2.4(4) gives the result. □

Lemma 3.1.3.3. *Let G act transitively on two irreducible varieties, X and Y, and let $\varphi : X \to Y$ be a bijective G-equivariant morphism. Then Y is complete if and only if X is complete.*

Proof. Theorem 4.3.3 of [9] tells us that, for any variety Z, aand any G-equivariant morphisms of homogeneous spaces $f : X_1 \to X_2$, the morphism $\varphi \times \mathrm{id} : X_1 \times Z \to X_2 \times Z$ is open. This result applied to our situation, combined with the fact that φ is bijective, tells us that $\varphi \times \mathrm{id}$ is in fact a homeomorphism. The result now follows immediately. □

Theorem 3.1.3.4 (Borel's Fixed Point Theorem). *Let B be a connected, solvable algebraic group, and X a (nonempty) complete variety on which B acts. Then B has a fixed point in X.*

Proof. We prove this by induction on $\dim B$. If $\dim B = 0$ then $B = \{e\}$ since it is connected, and the result is immediate. Suppose now that the theorem holds for solvable, connected groups of dimension less than n, and let $\dim B = n$. Consider $H = (B, B)$. Then by Proposition 2.2.1.3, H is connected, it is obviously solvable since B is, and Proposition 2.2.2.2 says that $\dim H < \dim B$. Therefore X^H is nonempty, call it Y. By Proposition 1.4.1.7(3), Y is closed in X, and so by Proposition 3.1.2.4(1), Y is complete. If $g \in B$ and $y \in Y$, then $g.y \in Y$. Indeed, for $h \in H$, $h(g.y) = g(h'.y) = g.y$ since H is normal in B, which shows that $g.y$ is a fixed point of the H-action. Therefore B stabilises Y.

By Proposition 1.4.3.3, there is an element $y \in Y$ such that the B-orbit of y is closed in Y. Note further that $H \subset B_y$, and so, given $h \in B_y$ and an arbitrary element $g \in B$,

$$(g^{-1}hg).y = h(h^{-1}g^{-1}hg).y = h.y = y$$

where the middle equality holds because $h^{-1}g^{-1}hg$ is a commutator, and so fixes y. Therefore B_y is normal in B. This means that B/B_y is an affine algebraic group, by Proposition 1.5.3.4. We consider the canonical map

$$\begin{aligned} \beta : B/B_y &\longrightarrow B.y \\ gB_y &\longmapsto g.y \end{aligned}$$

It is easy to check that this is a B-equivariant bijection. Since $B.y$ is closed in Y by construction, another application of Proposition 1.4.1.7(3) tells us that $B.y$ is complete. Therefore, by Lemma 3.1.3.3, B/B_y is complete. But we have already seen that this is an affine group, and so Proposition 3.1.2.4(6) says that it must be finite. But it is connected, since B is, and so $B = B_y$, which means y is a fixed point of the B-action on Y. But $y \in X$, and so it is in fact a fixed point of the B-action on X, as required. □

Note that this theorem gives a (really) quick proof of Theorem 2.2.5.1, namely the Lie-Kolchin Theorem:

Alternate Proof of Theorem 2.2.5.1. Since G is a closed, connected and solvable subgroup of $GL(V)$, then G acts on $\mathfrak{F}(V)$, flag variety of V, which Example 27 above says is complete. So G fixes a a point in $\mathfrak{F}(V)$, namely a flag

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

That is, G is triangular for a suitable choice of basis in V. □

3.1.4 Borel Subgroups

Definition 3.1.4.1. A *Borel subgroup* of G is a closed, connected solvable subgroup which is maximal, in the sense that it is properly included in no other.

Note that, if B is a closed connected, solvable subgroup of largest possible dimension in G, then it is a Borel subgroup. Indeed, if B is closed, connected, solvable and of largest possible dimension, then if B' is a closed, connected solvable subgroup with $B \subset B'$. Now, Proposition 3.2 of [6] says that the dimension of a proper, closed irreducible subset of an irreducible variety X has dimension strictly less that $\dim X$. This fact, when applied here, forces $B = B'$, and therefore B is maximal. The following result shows that all Borel subgroups of an algebraic group are of the same dimension.

Theorem 3.1.4.2. *Let B be any Borel subgroup of G. Then G/B is a projective variety, and all other Borel subgroups are conjugate to B and all conjugates of B are Borel subgroups.*

Proof. Suppose, firstly, that P is a conjugate of B, say $P = xBx^{-1}$ for some $x \in G$. Then P is certainly connected and solvable, since B is, and $\dim P = \dim B$. But this dimension is maximal, by definition, and so P is a Borel subgroup.

On the other hand, suppose P is a Borel subgroup. Since P is closed, Theorem 1.5.1.2 tells us that there is a rational representation $\varphi : G \to GL(V)$ and a 1-dimensional subspace $V_1 \subset V$ such that

$$P = \{g \in G \,|\, \varphi(g)V_1 = V_1\} \tag{3.4}$$

that is, P is the stabiliser of the action of G on V_1. Now P acts canonically on V/V_1, and since P is solvable, we have a full flag of subspaces of V/V_1 stable under P, by Lie-Kolchin, which we will denote as follows:

$$0 = W_1 \subset W_2 \subset \cdots \subset W_i \subset \cdots \subset W_n = V/V_1$$

We can take the direct sum of each of these subspaces with V_1 to get the following full flag:

$$0 \subset W_1 \oplus V_1 \subset W_2 \oplus V_1 \subset \cdots \subset W_i \oplus V_1 \subset \cdots \subset W_n \oplus V_1$$

But $W_n \oplus V_1 = V/V_1 \oplus V_1 \cong V$. We will denote by V_i the image of the restriction of this isomorphism to the subspace $W_i \oplus V_1$, noting that $W_1 \oplus V_1 = 0 \oplus V_1 \cong V_1$. We therefore have a full flag of subspaces of V:

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots \subset V$$

and this flag is stable under the action of P. Call this last flag $f \in \mathfrak{F}(V)$. Now G acts on $\mathfrak{F}(V)$ via φ, and we wish to consider the isotropy group of f, namely

$$G_f = \{g \in G \,|\, gf = f\}$$

Certainly we know that $P \subset G_f$ by the above argument. On the other hand, suppose $g \in G_f$. Then, in particular, $gV_1 = V_1$, whence equation (3.4) gives us $g \in P$, so we conclude $P = G_f$. It is not difficult to deduce that the map

$$G/P \longrightarrow G.f \subset \mathfrak{F}(V)$$

is bijective. Now since the isotropy group of any flag in $\mathfrak{F}(V)$ is itself solvable as it can by always be put in triangular form, the group P is of largest dimension among them. This then means that G/P is the smallest of all groups which are G quotiented by an isotropy group, and hence that $G.f$ is an orbit of minimal dimension. Proposition 1.4.3.3 then tells us that $G.f$ is closed, and so, being a closed subset of the projective variety $\mathfrak{F}(V)$ (see Example 12 in Chapter 1), it is complete. By Proposition 3.1.2.4 we then deduce that G/P is complete, and therefore that P is parabolic.

We still need to show that P is conjugate to the given Borel subgroup B. But we can define an action

$$\begin{aligned} B \times G/P &\longrightarrow G/P \\ (x, gP) &\longmapsto xgP \end{aligned}$$

by left multiplication. By Theorem 3.1.3.4, this action fixes a point $gP \in G/P$. That is, $xgP = gP$ for all $x \in B$, or $g^{-1}xgP = P$. Suppose now $x \in B$. Then $g^{-1}xg = g^{-1}xge \in g^{-1}xgP = P$. From this we conclude that $g^{-1}Bg \subset P$. But since each of these are Borel subgroups, both are maximal, and hence $g^{-1}Bg = P$. This proves the result. $\square$

Corollary 3.1.4.3. *The maximal tori (respectively, maximal connected unipotent subgroups) of G are also those of the Borel subgroups of G and they are all conjugate.*

Proof. Let T be a maximal torus of G. Then certainly T is solvable, and hence it lies inside some Borel subgroup B. As such, T is also a maximal torus of B, which is solvable, and we have already shown that, for solvable groups, all maximal tori are conjugate.

Suppose U is a maximal connected unipotent subgroup of G. Then U is connected and solvable, and so lies in some Borel subgroup B. Moreover, $U \subset B_u$, but, by maximality, this forces $U = B_u$. Suppose now U' is another maximal, connected unipotent subgroup. Then, by the same argument, $U = B'_u$ for some Borel subgroup B'. By Theorem 3.1.4.2, there exists an element $x \in G$ such that $B' = xBx^{-1}$, from which it is easy to see that $B_u = xB_ux^{-1}$, that is, $U' = xUx^{-1}$, as required. $\square$

An aside: there is an analogy that can be drawn between Borel subgroups and the Sylow p-subgroups of finite group theory. Indeed, Corollary 3.1.4.3 is analagous to the second Sylow Theorem, which says that, for a given prime, p, all Sylow p-subgroups are conjugate to each other.

Definition 3.1.4.4. Call the common dimension of the maximal tori of G the *rank* of G.

Proposition 3.1.4.5. *Let B be a Borel subgroup of G. Then G/B is the largest homogeneous space for G having the structure of a projective variety.*

Proof. Suppose H is a subgroup of G such that G/H is projective. Then B acts on G/H via left multiplication, and so, by Theorem 3.1.3.4, this action fixes a point $gH \in G/H$. Again it follows that $g^{-1}Bg \subset H$, whence $\dim G/H \leq \dim G/B$. $\square$

Corollary 3.1.4.6. *A closed subgroup of G is parabolic if and only if it includes a Borel subgroup. In particular, a connected subgroup H of G is a Borel subgroup if and only if H is solvable and G/H is projective.*

Proof. Suppose P is parabolic, and B a Borel subgroup. Then B acts via left multiplication on G/P, which is complete, and so Theorem 3.1.3.4 again shows us that there is an element $g \in G$ such that $g^{-1}Bg \subset P$. But $g^{-1}Bg$ is again Borel, and so P contains a Borel subgroup. On the other hand, supposing that P is a closed subgroup such that there is a Borel subgroup B with $B \subset P$, then we consider the surjective morphism $G/B \to G/P$. By Proposition 3.1.2.4(3), the image of this morphism, namely G/P, is complete, hence P is parabolic.

For the second statement, suppose H is solvable and G/H projective, then H is parabolic, and so contains a Borel subgroup B. But B is maximal, by definition, and so is properly included in no other solvable, connected group, from which we deduce $H = B$. On the other hand, supposing H is Borel, then H is certainly solvable, and Theorem 3.1.4.2 tells us G/H is projective. $\square$

Corollary 3.1.4.7. *Let $\varphi : G \to G'$ be an epimorphism of connected algebraic groups. Let H be a Borel subgroup (respectively parabolic subgroup, maximal torus, maximal connected unipotent subgroup) of G. Then $\varphi(H)$ is a subgroup of the same type in G', and all such subgroups of G' are obtained in this way.*

Proof. We begin by proving the result for Borel subgroups, so let $H = B$ be a Borel subgroup of G. We need to show that $B' = \varphi(B)$ is a Borel subgroup of G'. Consider the map

$$G \xrightarrow{\varphi} G' \longrightarrow G'/B'$$

We can define a map

$$\begin{aligned} \beta : G/B &\longrightarrow G'/B' \\ gB &\longmapsto \varphi(g)B' \end{aligned}$$

It is easy to check that β is a well defined homomorphism, and since φ is surjective, so is β. By Theorem 3.1.4.2, G/B is complete, and so Proposition 3.1.2.4(3) tells us that G'/B' is complete, whence B' is parabolic. Certainly $B' = \varphi(B)$ is solvable and connected, so the second part of Corollary 3.1.4.6 tells us that B' is Borel. Now, using Theorem 3.1.4.2 again, we know that all the other Borel subgroups of G' are conjugates of this B', and hence are images of Borel subgroups in G under φ. Indeed, supposing B_0 is another Borel subgroup of G', then there exists an element $h \in G'$ such that $B_0 = hB'h^{-1} = h\varphi(B)h^{-1}$. Now $h = \varphi(g)$ for some $g \in G$, so $B_0 = \varphi(gBg^{-1})$, and it is clear that gBg^{-1} is Borel in G.

Let H be a parabolic subgroup. By Corollary 3.1.4.6, H contains a Borel subgroup, B, say. Then, by the above argument, $\varphi(B)$ is a Borel subgroup which is certainly contained in $\varphi(H)$. Applying Corollary 3.1.4.6 again tells us $\varphi(H)$ is a parabolic subgroup.

Suppose now that H is a maximal connected unipotent subgroup. Then $H \subset B$ for some Borel subgroup, and $\varphi(H)$ is itself connected and unipotent and contained in the Borel subgroup $\varphi(B)$. Certainly, then

$$\varphi(H) \subset \varphi(B)_u = \varphi(B_u) = \varphi(H),$$

and so $\varphi(H) = \varphi(B)_u$ is a maximal unipotent connected subgroup of G'.

Suppose, finally, that H is a maximal torus. Again, we take $H \subset B$ for some Borel subgroup, and note by Theorem 2.3.4.4 that $B = H \ltimes B_u$. Therefore $\varphi(B) = \varphi(H) \cdot \varphi(B_u)$. Certainly $\varphi(H)$ is a torus, since it is connected, commutative since φ is surjective, and consists of semisimple elements. Additionally, it is immediate that $\varphi(H) \cap \varphi(B_u) = \{e\}$, and so $\varphi(B) = \varphi(H) \ltimes \varphi(B_u)$.

Now we have already seen that $\varphi(B)$ is a Borel subgroup, and that $\varphi(B_u)$ is a maximal unipotent subgroup of $\varphi(B)$. Applying Theorem 2.3.4.4 again, we have $\varphi(B) = T' \ltimes \varphi(B_u)$ for some maximal torus T' of $\varphi(B)$. But this implies $\dim T' = \dim \varphi(B) - \dim \varphi(B_u) = \varphi(H)$, which means that $\varphi(H)$ is also a maximal torus. □

Example 28. Recall that $\mathrm{PGL}(2, K) = GL(2, K)/S$, where S denotes the group of scalar matrices $S = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = K^* = \mathbb{G}_m$. Consider the quotient morphism

$$\alpha : GL(2, K) \longrightarrow \mathrm{PGL}(2, K)$$

Note firstly that α shows us that $\dim \mathrm{PGL}(2, K) = 3$, and that $\mathrm{PGL}(2, K)$ is connected. Now, it is fairly obvious that $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is a Borel subgroup of $GL(2, K)$. We will discuss this in more detail in Chapter 5, but for now this means Corollary 3.1.4.7 tells us that B/S is a Borel subgroup of $\mathrm{PGL}(2, K)$. Similarly, $T = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ is a maximal torus of $GL(2, K)$, and so Corollary 3.1.4.7 tells us that T/S is a maximal torus of $\mathrm{PGL}(2, K)$. Note further that the morphism of algebraic groups

$$\begin{aligned} f : K^* &\longrightarrow T/S \\ a &\longmapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is an isomorphism. It follows that $\dim T/S = \mathrm{rank}(\mathrm{PGL}(2, K)) = 1$.

Proposition 3.1.4.8. *An automorphism σ of G which fixes all elements of a Borel subgroup B must be the identity.*

Proof. Construct a morphism

$$\begin{aligned} \varphi : G &\longrightarrow G \\ x &\longmapsto \sigma(x)x^{-1} \end{aligned}$$

Now $\varphi(B) = \{e\}$, and so φ factors through G/B. But since G/B is complete, Proposition 3.1.2.4(3) implies that $\varphi(G)$ is closed and complete. But $\varphi(G)$ is also affine, being a closed subset of an algebraic group, and therefore $\varphi(G) = \{e\}$ by Proposition 3.1.2.4(6). □

Corollary 3.1.4.9. *Let B be a Borel subgroup of a connected algebraic group G. Then $Z(G)^\circ \subset Z(B) \subset C_G(B) = Z(G)$.*

Proof. Certainly $Z(G)^\circ$ is solvable, being commutative, and closed and connected, being an identity component. Therefore it is contained in some Borel subgroup B'. By Theorem 3.1.4.2, there exists an element $x \in G$ such that $B' = xBx^{-1}$. Then, for any element $g \in Z(G)^\circ$, we have $g \in B' = xBx^{-1}$, and so $g = xbx^{-1}$ for some $b \in B$. But then $b = x^{-1}gx = g$, since g is central, and therefore $Z(G)^\circ \subset B$. In particular, $Z(G)^\circ \subset Z(B)$.

Now let $x \in Z(B)$. Then $xBx^{-1} = Bxx^{-1} = B$, so $x \in C_G(B)$. Similarly, if $x \in Z(G)$, then $xBx^{-1} = Bxx^{-1} = B$, so $x \in C_G(B)$. We have shown, then, that $Z(B) \subset C_G(B)$ and $Z(G) \subset C_B(G)$. It only remains to show that $C_G(B) \subset Z(G)$. But, if $x \in C_G(B)$, then the map $\sigma = \mathrm{Int}(x)$, an automorphism of G, fixes B, and so Proposition 3.1.4.8 says $\mathrm{Int}(x) = \mathrm{id}_G$, therefore $x \in Z(G)$, as required. □

3.1.5 Cartan Subgroups

Proposition 3.1.5.1. *Let G be a connected algebraic group, with B a Borel subgroup of G.*

1. *If B is nilpotent then $G = B$.*

2. *G is nilpotent if and only if G has just one maximal torus.*

Proof. We prove (1) by induction on the dimension of G. Certainly, if $\dim G = 0$, then $G = B = \{e\}$. Before we start the induction step, note that if $B = \{e\}$, then $G = G/B$ is complete. But G is affine, being an algebraic group, and so Proposition 3.1.2.4(6) tells us that $G = \{e\} = B$. Suppose, then, that $B \neq \{e\}$. Then, since B is nilpotent, we have $Z(B) \neq \{e\}$. Corollary 3.1.4.9 now tells us that $Z(G) \neq \{e\}$, or $\dim Z(G)^\circ = \dim Z(G) > 0$. Now $\dim G > \dim G/Z(G)^\circ$, and so, by induction, $G/Z(G)^\circ = B/Z(G)^\circ$, the latter being a nilpotent Borel subgroup by Corollary 3.1.4.7. By comparison of dimension, this is enough to imply that $G = B$.

For (2), one direction is already done: Corollary 2.3.4.3 told us that a connected nilpotent group has a unique maximal torus. For the reverse direction, suppose T is the only maximal torus of G. Since T is connected and solvable, it is contained in some Borel subgroup B, and Corollary 3.1.4.3 tells us that T is the only maximal torus of B. Corollary 2.3.4.6 tells us that each semisimple element of B lies in this torus T, and so $B_s = T$, and thus is a subgroup. Proposition 2.3.4.2 then implies that B is nilpotent, whereupon we can apply (1), which tells us $B = G$, and in particular G is nilpotent, as required. □

Corollary 3.1.5.2. *Let G be a connected algebraic group G with* $\dim \leq 2$. *Then G is solvable.*

Proof. If $\dim G = 0$ then $G = \{e\}$ and the result is trivial. Suppose, then $1 \leq \dim G \leq 2$. Let T be a maximal torus of G. Suppose, for a contradiction, that G not solvable. Then certainly $T \subsetneq G$, and since both G and T are connected, we know that $\dim T < \dim G$. If $\dim G = 1$, then $T = \{e\}$ and so by Proposition 3.1.5.1 (2) tells us that G is nilpotent and so solvable. Similarly, if $\dim G = 2$ then either $\dim T = 0$, and we are again in the case just mentioned, or $\dim T = 1$. But we know that there are only two connected algebraic groups of dimension one, namely $\mathbb{G}_a$ and $\mathbb{G}_m$, and only one of these is solvable, namely $\mathbb{G}_m$. It follows that, again, G has a unique maximal torus, and so another application of Proposition 3.1.5.1 (2) tells us that G is nilpotent and so solvable. $\square$

Definition 3.1.5.3. The connected centraliser C of a maximal torus T of G (that is, $C = C_G(T)^\circ$) is called a *Cartan subgroup* of G.

Lemma 3.1.5.4. *The Cartan subgroups of an algebraic group G are conjugate to each other.*

Proof. Let T, T' be two maximal tori of G. Then, by Corollary 3.1.4.3, we have $T' = xTx^{-1}$ for some $x \in G$. Write $C = C_G(T), C' = C_G(T')$. Then $C' = xCx^{-1}$. Indeed, suppose $g \in C'$, then

$$x^{-1}gxTx^{-1}g^{-1}x = x^{-1}gT'g^{-1}x = x^{-1}T'x = T$$

so $x^{-1}gx \in C$, or $C' \subset xCx^{-1}$. On the other hand, if $g \in C$, then

$$xgx^{-1}T'xg^{-1}x^{-1} = xgTg^{-1}x^{-1} = xTx^{-1} = T'$$

and so, $xgx^{-1} \in C'$, or $xCx^{-1} \subset C'$. Hence $xCx^{-1} = C'$, as required. $\square$

Corollary 3.1.5.5. *Let T be a maximal torus of G, and $C = C_G(T)^\circ$. Then C is nilpotent, and $C = N_G(C)^\circ$.*

Proof. Clearly $T \subset C$, is a maximal torus of C. Since maximal tori are conjugate, by Corollary 3.1.4.3, this implies that T is the unique maximal torus of C Indeed, if T' is a torus of C, then $T' = xTx^{-1}$ for some $x \in C$, hence $T' = xx^{-1}T = T$. Now we apply Proposition 3.1.5.1(2), which tell us that C is nilpotent.

We aim to show now that $N_G(T) = N_G(C)$. Let $x \in N_G(T)$, with $g \in C, t \in T$. Then

$$t(xgx^{-1})t^{-1} = x(x^{-1}tx)g(x^{-1}t^{-1}x)x^{-1} = xgx^{-1} \tag{3.5}$$

where the second equality holds because $x^{-1}tx \in T$, and so commutes with g. Equation (3.5) shows that $xgx^{-1} \in C$, and so $x \in N_G(C)$, whence $N_G(T) \subset N_G(C)$. For the reverse inclusion, let $x \in N_G(C)$. Since $T \subset C$, we have $xTx^{-1} \subset xCx^{-1} = C$, which implies xTx^{-1} is a maximal torus of C, and so by the above argument, $xTx^{-1} = T$, which implies $x \in N_G(T)$. Therefore $N_G(C) \subset N_G(T)$, as required.

Putting all this together gives

$$C = C_G(T)^\circ = N_G(T)^\circ = N_G(C)^\circ$$

where the middle equality is by Corollary 2.1.6.3. $\square$

3.1.6 The Density Theorem

Lemma 3.1.6.1. *Let H be a closed, connected subgroup of G, and set*

$$X = \bigcup_{x \in G} xHx^{-1}$$

1. *If G/H is complete, then X is closed.*
2. *If there is an element $h \in H$ such that the fixed point $(G/H)^h$ is finite, then X contains a nonempty open subset of G, and so is dense.*

Proof. Define an isomorphism φ as follows:

$$\begin{aligned} \varphi : G \times G &\longrightarrow G \times G \\ (x, y) &\longmapsto (x, xyx^{-1}) \end{aligned}$$

Since $G \times H$ is closed in $G \times G$, then $\varphi(G \times H)$ is closed in $G \times G$, by the continuity of φ^{-1}. Now consider the quotient morphism

$$\begin{aligned} \psi : G \times G &\longrightarrow (G \times G)/(H \times \{e\}) \\ (x, y) &\longmapsto (xH, y) \end{aligned}$$

We saw in Theorem 1.5.3.2 that ψ is a surjective, open morphism. Note also that Lemma 1.5.3.5 tells us that the image of ψ is equal to $G/H \times G$. We define a set $M = \psi \circ \varphi(G \times H)$. That is,

$$M = \{(xH, y) \mid (x, y) \in \varphi(G, \times H)\} = \{(xH, y) \mid y = xhx^{-1} \text{ for some } h \in H\}$$

Note also that $X = \mathrm{pr}_2(M)$.

We first show that $\varphi(G \times H) = \psi^{-1}(M) = \psi^{-1}(\psi \circ \varphi(G \times H))$. The inclusion $\varphi(G \times H) \subset \psi^{-1}(M)$ is immediate, so suppose on the other hand that $(x, y) \in \psi^{-1}(M)$. That is, $(xH, y) \in M$, which is to say $y = xhx^{-1}$ for some $h \in H$. Then $(x, y) = \varphi(x, h)$, and so $\psi^{-1}(M) \subset \varphi(G \times H)$, as required. Now $G \times H$ is closed, and so $\varphi(G \times H)$ is closed since φ is an isomorphism. Therefore $\psi^{-1}(M)$ is closed, too.

Since $\psi^{-1}(M)$ is closed, its complement $\psi^{-1}(M)^c$ is open. Now, ψ is an open map, so $\psi(\psi^{-1}(M)^c)$ is open. We claim that this latter set is in fact equal to M^c. Indeed, if $b \in \psi(\psi^{-1}(M)^c)$, then $b = \psi(a)$ for some $a \in \psi^{-1}(M)^c$. That is, $\psi(a) \notin M$, and so $\psi(\psi^{-1}(M)^c) \subset M^c$. On the other hand, if $b \in M^c$, then $b = \psi(a)$ for some $a \notin \psi^{-1}(M)$, since ψ is surjective. Therefore $b \in \psi(\psi^{-1}(M)^c)$, and so $M^c \subset \psi(\psi^{-1}(M)^c)$. Thus M^c is an open set, and so M is closed.

We are assuming that G/H is complete, and so $\mathrm{pr}_2(M)$ is closed. But

$$\begin{aligned} \mathrm{pr}_2(M) &= \{y \in G \mid y = xhx^{-1}, \text{ for some } x \in G, h \in H\} \\ &= \{xhx^{-1} \mid x \in G, h \in H\} = X \end{aligned}$$

and so we have shown that X is closed, and so (1) is proven.

We turn to (2). This statement relies on some results on morphisms and dimension. In particular, we use Theorem 4.3 of [6] which says that, given a surjective morphism of (smooth) irreducible varieties $f : X_1 \to X_2$, then there is a fibre $f^{-1}(x)$ for some $x \in X_2$ such that $\dim f^{-1}(x) + \dim X_1 = \dim X_1$. Indeed, in the first place, since $\mathrm{pr}_1 : M \times G \to G/H$ is surjective, then another application of the now familiar Theorem 4.3 of [6] says that $\dim M = \dim G/H + \dim \mathrm{pr}_1^{-1}(xH)$ for some $x \in G$. But $\mathrm{pr}_1^{-1}(xH) \cong xHx^{-1}$, and so $\dim M = \dim G/H + \dim H = \dim G$. On the other hand, $\mathrm{pr}_2 : M \to X$, is characterised by the fact that $\dim \mathrm{pr}_2^{-1}(h)$ is zero, and another application of Theorem 4.1 of [6] then tells us that $\dim M = \dim \bar{X}$, since both M and therefore $\bar{X}$ are irreducible varieties and the projection map is surjective. Therefore $\bar{X} = G$, and so X is dense.

However, X is also constructible, since $X = \mathrm{pr}_2 \circ \psi \circ \varphi(G \times G)$. Lemma 1.3.1.8 then says that X contains a nonempty subset U which is open in $\bar{X} = G$, as required. □

Recall the notation of Definition 2.1.1.3: given an algebraic group G, denote by G_u the subset of G consisting of its unipotent elements, and denote by G_s the subset of G consisting of its semisimple elements.

Theorem 3.1.6.2 (The Density Theorem). *Let B be a Borel subgroup, T a maximal torus of G, with $C = C_G(T)^\circ$ the associated Cartan subgroup. Then*

1. *The union of the conjugates of C contains a dense open subset of G.*
2. *The union of all conjugates of B is G.*
3. *The union of all conjugates of T is G_s.*
4. *The union of all conjugates of B_u is G_u.*

Proof. We begin by proving (1). Proposition 2.3.1.8 tells us that there is a semisimple element $t \in T$ such that

$$C = C_G(t)^\circ$$

We claim that, with $C = H$, this element t satisfies the conditions of Lemma 3.1.6.1 (2). Indeed, suppose t fixes an element xC of G/H. That is, $txC = xC$, or $x^{-1}tx \in C$. Since $t \in T$, it is semisimple, and therefore so is $x^{-1}tx$. So we have $x^{-1}tx \in C_s = T$. Certainly this is enough to tell us that $T \subset C_G(x^{-1}tx)^\circ$ since the elements of T commute. We next observe that $C_G(x^{-1}tx) = x^{-1}C_G(t)x$. For, suppose $g \in C_G(x^{-1}tx)$, so $(x^{-1}tx)g = g(x^{-1}tx)$. Then

$$(xgx^{-1})t = x(gx^{-1}tx)x^{-1} = x(x^{-1}txg)x^{-1} = t(xgx^{-1})$$

so $xgx^{-1} \in C_G(t)$, whence $C_G(x^{-1}tx) \subset x^{-1}C_G(t)x$. On the other hand, suppose $g \in C_G(t)$, so $gt = tg$. Then

$$(x^{-1}gx)(x^{-1}tx) = x^{-1}gtx = x^{-1}tgx = (x^{-1}tx)(x^{-1}gx)$$

whence $x^{-1}gx \in C_G(x^{-1}tx)$, or $x^{-1}C_G(t)x \subset C_G(x^{-1}tx)$, as required. So far we have

$$T \subset C_G(x^{-1}tx)^\circ = [x^{-1}C_G(t)x]^\circ = x^{-1}C_G(t)^\circ x = x^{-1}Cx \tag{3.6}$$

that is, $xTx^{-1} \subset C$. But xTx^{-1} is itself a maximal torus, and thus a maximal torus of C. But C is nilpotent and therefore has a unique maximal torus, and so we have $xTx^{-1} = T$. Now, $xC_ux^{-1} = C_u$ since conjugation won't turn a unipotent element into a semisimple one, and so the equation $C = T \times C_u$ tells us $xCx^{-1} = C$, or $x \in N_G(C)$. By Corollary 3.1.5.5, $C = N_G(C)^\circ$, and so, for all the elements $x \in N_G(C)^\circ$, the coset $xC = C$. The only other distinct cosets which are fixed by t then are those xC for which $x \in N_G(C) - N_G(C)^\circ$. But since $[N_G(C) : N_G(C)^\circ]$ is finite, we only have finitely many such cosets, and therefore only finitely many elements of G/C which are fixed by t.

This means that the requirements of Lemma 3.1.6.1 (2) are satisfied, and so the union of conjugates of Cartan subgroups

$$X = \bigcup_{x \in G} xCx^{-1}$$

contains a dense open subset of G, which proves (1).

For (2), note that since C is closed, connected and nilpotent, and therefore solvable, it must be contained in some Borel subgroup, B, because B is maximal under these conditions. It follows immediately that, for

$$X' = \bigcup_{x \in G} xBx^{-1}$$

we have $X \subset X'$, and so X' contains a dense open subset of G, from which we immediately get $\overline{X'} = G$. But B is complete by Theorem 3.1.4.2, and so by Lemma 3.1.6.1 (1), X' is closed, and so $X' = G$, as required.

For (3), use Proposition 2.3.4.7. Certainly, if x lies in some torus, it is semisimple, and so lies in G_s. On the other hand, suppose that $x \in G_s$. Then $x \in yBy^{-1}$ for some $y \in G$ by (2), and so, by Proposition 2.3.4.7 applied to the Borel subgroup yBy^{-1} illustrates that x lies in some maximal torus, all of which are conjugate by Corollary 3.1.4.3.

Finally, for (4), suppose firstly that $x \in yB_uy^{-1}$. Then x is a conjugate of a unipotent element, and so is itself unipotent, thus lies in G_u. On the other hand, suppose $x \in G_u$. Then (2) guarantees that $x \in yBy^{-1}$ for some $y \in G$. But x is unipotent, and so $x \in yB_uy^{-1}$, as required. □

Corollary 3.1.6.3. *1. Each semisimple element of G lies in a maximal torus.*

2. Each unipotent element of G lies in a connected nilpotent subgroup of G.

Proof. (1) follows at once from Theorem 3.1.6.2 (3). For (2), note that each B_u has a unique maximal torus, namely $\{e\}$. By 3.1.5.5, it follows that each B_u is nilpotent, and since Theorem 3.1.6.2 (4) tells us that each unipotent element of G lies in one of these connected, nilpotent subgroups, this proves the result. □

Corollary 3.1.6.4. *If B is any Borel subgroup of G then $Z(G) = Z(B)$.*

Proof. By Corollary 3.1.4.9, we already know $Z(B) \subset Z(G)$. Suppose, on the other hand, $x \in Z(G)$. Theorem 3.1.6.2 (2) tells us that x lies in some conjugate of B, say $x \in yBy^{-1}$. Then $y^{-1}xy \in B$. But x is central, so $y^{-1}xy = x$, and therefore $x \in Z(G) \cap B = Z(B)$, so $Z(G) \subset Z(B)$, as required. □

Theorem 3.1.6.5. *Let S be a torus in G. Then $C_G(S)$ is connected.*

Proof. This has already been proven by Propostion 2.3.4.7 in the case where G is solvable, since S certainly consists of semisimple elements, being isomorphic to $D(n, K)$ for some n.

We need to show that $C_G(S) \subset C_G(S)^\circ$, so let $x \in C_G(S)$. Certainly, since S is a torus, it lies inside some Borel subgroup B. Now, by Theorem 3.1.6.2(4), we know that $x \in yBy^{-1}$ for some $y \in G$. That is, $y^{-1}xy \in B$, or $xyB = yB$. Define X to be the fixed point set of x in G/B, that is

$$X = \{gB \in G/B \,|\, xgB = gB\}$$

From above we see that $yB \in X$, and so it is nonempty, and then we know by Proposition 1.4.1.7(3) that X is closed in G/B. But G/B is complete, and so X is, too. Now, suppose $s \in S$, and $gB \in X$. Then

$$x(sgB) = s(xgB) = sgB$$

where the first equality hold because $x \in C_G(S)$, and so s commutes with x. This then shows that $sgB \in X$, and so S leaves X stable. We have show, then, that S is a connected, solvable group which acts on the complete variety X. Therefore can then apply Theorem 3.1.3.4 to show that S must have a fixed point in X, that is, there is an element $hB \in X$, such that $shB = hB$ for all $s \in S$. Defining $B' = hBh^{-1}$, we see at once that $s \in B'$. But s was an arbitrary element of S, and so $S \subset B'$. Moreover, since $hB \in X$, we have $xhB = hB$, or $x \in hBh^{-1} = B'$.

Certainly, then $x \in C_{B'}(S) = C_G(S) \cap B'$. But $C_{B'}(S) = C_{B'}(S)^\circ$ by another application of Proposition 2.3.4.7. Moreover, since $C_{B'}(S) \subset C_G(S)$, it follows that $C_{B'}(S)^\circ \subset C_G(S)^\circ$, by Lemma 1.3.2.2. But then

$$x \in C_{B'}(S) = C_{B'}(S)^\circ \subset C_G(S)^\circ$$

and since x was an arbitrary element of $C_G(S)$, it follows that $C_G(S) \subset C_G(S)^\circ$, which is enough to show that $C_G(S)$ is connected. □

Corollary 3.1.6.6. *The Cartan subgroups of G are precisely the centralisers of maximal tori. That is,*

$$C_G(T)^\circ = C_G(T)$$

for all maximal tori, T. □

Theorem 3.1.6.7. *Let S be a torus in G, with $C = C_G(S)$, and B a Borel subgroup of G such that $S \subset B$. Let X be the set of fixed points in G/B, and Y any connected component of X. Then C acts transitively on Y.*

Proof. Clearly $B \in X$, and we assume that $B \in Y$. We must first show that C acts on Y, before we show that this action is transitive. Certainly, the torus S acts on the variety G/B. If X is the fixed point set of this action, then it is easy to see that C acts on X, since, given $xB \in X$, for any element $g \in C$, we have $gxB \in X$. Indeed, given an element $s \in S$, then $s(gxB) = g(sxB) = gxB$, and so gxB is fixed by S, as required. Now C is connected by Theorem 3.1.6.5, so it stabilises each connected component Y of X by Proposition 1.4.1.7(4). We therefore have a C-action on Y.

To show that this action is transitive, it suffices to show that $CB = Z$, where Z is the inverse image in G of Y under the quotient morphism $\alpha : G \to G/B$. For, suppose $xB \in Y$, we wish to show that $C.(xb) = Y$. So, given an arbitrary element $yB \in Y$, we note that $x, y \in Z$, and so $x = gb, y = g'b'$, for some $g, g' \in C, b, b' \in B$. In this case, then, $xB = gbB = gB$, and similarly, $yB = g'B$. So, we have

$$yB = g'B = g'(g^{-1}g)B = (g'g^{-1})xB$$

and certainly $g'g^{-1} \in C$. It remains to show, then, that $CB = Z$.

Since $Z = \alpha^{-1}(Y)$, it is connected. Suppose now $s \in S, z \in Z$. Then $\alpha(z) = zB \in Y$, and so $szB = zB$, because elements of Y are fixed by S, by definition. This then means $z^{-1}sz \in B$, for any $s \in S, z \in Z$. Thus we can define a map

$$\begin{aligned} \psi : Z \times S &\longrightarrow B \\ (z, s) &\longmapsto z^{-1}sz \end{aligned}$$

If we compose this with the canonical map $B \to B/B_u$, we get a map

$$\varphi : Z \times S \to B/B_u$$

Note that both S and B/B_u are tori, and Z is connected, and so they satisfy the hypotheses of Proposition 2.1.6.2, which tells us that, the map $z \mapsto \varphi(z, \cdot)$ is constant, that is, $z^{-1}szB_u = z'^{-1}sz'B_u$ for any $z, z' \in Z$, $s \in S$. In particular, since we have assumed that $B \in Y$, we have $e \in Z$, and so $z^{-1}szB = sB_u$. This in turn tells us that $z^{-1}Sz \subset SB_u$. Now, S and $z^{-1}Sz$ are both tori of SB_u, indeed they are clearly maximal tori. Since SB_u is solvable and connected, we can apply Corollary 3.1.4.3 which tells us that all maximal tori are conjugate. Thus, there is an element $u \in B_u$ such that $u^{-1}z^{-1}Szu = S$, whence $zu \in N = N_G(S)$. In turn, we see that $z \in Nu^{-1} \subset NB$. On the other hand, given any element $g \in C$, we know by the fact that $B \in Y$ and that the C-action stabilises Y, that $gB \in Y$, and so $CB \subset Z$. We therefore have a chain $CB \subset Z \subset NB$. By our Corollary 2.1.6.3, however, we know that $N^\circ = C^\circ$, and so $[N : C]$ is finite. This in turn tells us that $\dim C = \dim N$, and moreover $\dim CB = \dim NB$, and therefore both are equal to $\dim Z$. Now, since both CB and Z are connected, we have $CB = Z$, as required. □

Corollary 3.1.6.8. *Let S, C and B be as in Theorem 3.1.6.7. Then $C_B(C) = C \cap B$ is a Borel subgroup of C, and all Borel subgroups of C are obtained in this way.*

Proof. If X is, as above, the fixed point set of the S-action on G/B, then we know that X is a closed subvariety of G/B, and therefore complete by Proposition 3.1.2.4(1). Moreover, if Y is a connected component of X, which contains B, then Y itself is closed, and so is complete. In Theorem 3.1.6.7, we showed that $Y = CB$, where $\alpha : G \to G/B$ is the canonical quotient morphism. We can therefore construct a G-equivariant morphism

$$\begin{aligned} \beta : C/C \cap B &\longrightarrow Y = CB \\ g(C \cap B) &\longmapsto gB \end{aligned}$$

We claim that β is an isomorphism. It is certainly well defined, since if $h^{-1}g \in C \cap B$ for $h, g \in C$, then $h^{-1}g \in B$. It is injective, since if $h^{-1}g \in B$ for $h, g \in C$, then $h^{-1}g \in C \cap B$. Finally, it is surjective, since any $yB \in B$ can be expressed as $yB = gB$ with $g \in C$, and so $yB = \beta(g)$.

Certainly C acts on $C/(C \cap B)$ transitively, and Theorem 3.1.6.7 told us the C-action on Y is transitive. We can therefore apply Lemma 3.1.3.3 to β to show that, since Y is complete, as explained above, then $C/C \cap B$ is complete, which is to say that the group $C \cap B$ is projective in C. Now Corollary 3.1.4.6 tells us that $C \cap B$ contains a Borel subgroup B' of C. But $C \cap B$ itself is connected and solvable, and therefore, since B' is maximal with these properties, then $B' = C \cap B$ is a Borel subgroup of C. □

3.1.7 The Normaliser Theorem

In this section, G denotes a *connected* algebraic group. The aim is to prove that, if B a Borel subgroup of G, then $B = N_G(B)$.

Lemma 3.1.7.1. *If B is a Borel subgroup of G, then $B = N_G(B)^\circ$.*

Proof. Write $N = N_G(B)$. Certainly $B \subset N^\circ$, and therefore B is a Borel subgroup of N°. Now, for all $x \in N^\circ$, by definition we have $xBx^{-1} = B$. By Theorem 3.1.4.2, all Borel subgroups of N° are conjugates of B, and therefore B is the unique Borel subgroup of N°. But Theorem 3.1.6.2 tells us that N° is the union of all its Borel subgroups, and therefore $B = N^\circ$, as required. □

Lemma 3.1.7.2. *The radical $R(G)$ of G is the identity component of the intersection of all Borel subgroups. Similarly, the unipotent radical $R_u(G)$ of G is the identity component of the intersection of all the unipotent parts of the Borel subgroups.*

Proof. Let A be the connected component of the intersection of all Borel subgroups of G. Then A is solvable, since it is a subgroup of a Borel group. It is connected by definition, and we also need to show that it is normal, so suppose $x \in G$. We wish to show that $xAx^{-1} = A$, and it therefore suffices to show that, given an arbitrary Borel subgroup B, we have $xAx^{-1} \subset B$. Let $g \in A$. Then $g \in x^{-1}Bx$ since the latter is a Borel subgroup. Therefore

$xgx^{-1} \in x(x^{-1}Bx)x^{-1} = B$, as required. So A is normal, connected and solvable, and therefore is contained in $R(G)$, which is defined to be maximal under these conditions.

For the reverse inclusion, we simply need to show that $R(G)$ is contained in every Borel subgroup of G. So let B be an arbitrary Borel subgroup. Certainly, since $R(G)$ is connected and solvable, it lies in some Borel subgroup B', which itself is conjugate to B, say $B' = xBx^{-1}$ for $x \in G$. Then, since $R(G)$ is normal in G, we have

$$R(G) = xR(G)x^{-1} \subset xB'x^{-1} = B$$

Since B was arbitrary, it follows that $R(G)$ lies inside every Borel subgroup, and, being connected, must therefore lie inside A, as required.

We can prove the statement about $R_u(G)$ similarly. Define a group

$$U = \Big(\bigcap_{B \in \mathscr{B}} B_u \Big)^{\circ}$$

U is clearly unipotent, and we can show in a similar fashion to above that it is normal. To do so, it suffices to show that, for any $B \in \mathscr{B}$, and any $x \in G$, we have $xUx^{-1} \subset B_u$. Consider $u \in U$. Now $B' = x^{-1}Bx$ is a Borel subgroup, and therefore $u \in B'_u = (x^{-1}Bx)_u = x^{-1}B_u x$, the latter equality being easy to verify by a quick calculation. Then $xux^{-1} \in B_u$, and therefore $xUx^{-1} \subset U$, that is, U is normal. We know from Proposition 3.1.1.7 that $R_u(G)$ is the largest connected, normal unipotent subgroup of G. Therefore $U \subset R_u(G)$.

For the reverse inclusion, it suffices to show that $R_u(G)$ is contained in the unipotent part of every Borel subgroup, for which we can apply the first part of the proposition. But from above we saw that $R(G) \subset B$ for any Borel group B. Therefore, $R_u(G) = (R(G))_u \subset B_u$, and we are done. □

Theorem 3.1.7.3 (The Normaliser Theorem). *Let B be a Borel subgroup of G. Then $B = N_G(B)$.*

Proof. We will prove the result by induction on $\dim G$. The theorem is obvious in case $\dim G = 0$, that is, $G = \{e\}$, since G is assumed to be connected. Assume then that $\dim G = n$ and that the result holds for groups of dimension less than n. We divide the induction step up into two cases: firstly, the case where G is semisimple, that is, where $R(G)$ is trivial. We then tackle the general case.

Suppose G is semisimple, and let $x \in N = N_G(B)$. We wish to show that $x \in B$. Let T be a maximal torus of G such that $T \subset B$. Then $xTx^{-1} \subset B$, and this is again a maximal torus. By Corollary 3.1.4.3, these two maximal tori of B are conjugate, so there is an element $y \in B$ such that $yxTx^{-1}y^{-1} = T$. Certainly it is true that $yx \in B$ if and only if $x \in B$, and so we can replace yx with x, and assume that our arbitrary element x of N also lies in $N_G(T)$, that is $xTx^{-1} = T$. Now define a group $S = C_T(x)^{\circ}$. This is a connected subgroup of the torus T, and so is again a torus, by Theorem 2.1.5.5, say.

Suppose, on the one hand, that $S \neq \{e\}$. Then, since S is a torus and therefore commutative, it follows that $S \subset C = C_G(S)$ and so C is nontrivial.

Moreover, since S is solvable and connected, and normal in C, it is contained in the radical $R(C)$, and so this radical, too, is nontrivial. Thus C cannot equal G, which is semisimple by assumption, and so $C \subsetneq G$. Moreover, by Theorem 3.1.6.5, C is connected, since S is a torus, and so $\dim C < \dim G$. Now, Corollary 3.1.6.8 tells us that $B' = C \cap B$ is a Borel subgroup of C, and so, by our induction hypothesis, we have $N_C(B') = B'$. We know $xBx^{-1} = B$, since $x \in N$, and $xCx^{-1} = C$ since $x \in S \subset C$. This tells us that $xB'x^{-1} = B'$, and so $x \in N_C(B') = B' \subset B$, and we are done.

On the other hand, suppose $S = \{e\}$. We first consider the commutator morphism

$$\begin{aligned} \gamma_x : T &\longrightarrow T \\ t &\longmapsto txt^{-1}x^{-1} \end{aligned}$$

Note that the image of γ_t does indeed lie in T, since $x \in N_G(T)$. Moreover, given $s, t \in T$, we have

$$\begin{aligned} \gamma_x(st) &= stxt^{-1}s^{-1}x^{-1} = s(t)(xt^{-1}x^{-1})(xs^{-1}x^{-1}) \\ &= s(xs^{-1}x^{-1})(t)(xt^{-1}x^{-1}) = (\gamma_x(s))(\gamma_x(t)) \end{aligned}$$

Therefore, γ_x is a group homomorphism. Clearly, $\ker \gamma_x = C_T(x)$. Moreover, since $S = \{e\}$, and $[C_T(x) : S]$ is finite, we conclude that $C_T(x)$ itself is finite. Proposition 1.3.2.1 told us that $\dim T = \dim(\ker \gamma_x) + \dim(\mathrm{im}\gamma_x)$, whence we have $\dim T = \dim(\mathrm{im}\gamma_x)$, and since T and $\mathrm{im}\gamma_x$ are both connected, this implies $T = \mathrm{im}\gamma_x$, which is to say, γ_x is surjective.

Define the group M to be that generated by B and the element x. This is clearly a subgroup of N, since $x \in N$ and $B \subset N$. Moreover, M is closed in G, since B and $\{x\}$ are. Since T is surjective, we have $T \subset (M, M)$. Consequently, since $B = TB_u$ by Theorem 2.3.4.4, it is clear that B lies in the subgroup of M which is generated by (M, M) and B_u.

Theorem 1.5.1.2 tells us that we can find a rational representation $\rho : G \to GL(V)$, with a one-dimensional subspace L of V such that

$$M = \{g \in G \mid \rho(x)L = L\} \tag{3.7}$$

There is also a character χ associated to this representation, $\chi : M \to \mathbb{G}_m$, where $\chi(g)v = \rho(g)v$ for some nonzero $v \in L$. Now $\chi(x_u) = \chi(x)_u = e$, since the only unipotent element of $\mathbb{G}_m$ is the identity. Therefore $\chi(B_u) = \{e\}$. Moreover, since (M, M) is generated by commutators, the image of which in $\mathbb{G}_m$ cancel due to commutativity in $\mathbb{G}_m$, we also have $\chi\big((M, M)\big) = \{e\}$. Since B is contained in the group generated by B_u and (M, M), we have $\chi(B) = \{e\}$.

Let $v \in L$ be a nonzero vector, and let Y be the orbit of v under $\rho(G)$, that is,

$$Y = \{\rho(g)v \mid G \in G\} \subset V$$

We can now define a surjective morphism

$$\begin{aligned} \mu : G/B &\longrightarrow Y \\ gB &\longrightarrow \rho(g)v \end{aligned}$$

This is well defined since, if $gB = hB$ then $h^{-1}g \in B$, and so $\rho(h^{-1}g) = \rho(h)^{-1}\rho(g) = e$, whence $\mu(gB) = \mu(hB)$. Now G/B is complete since B is Borel, and so Proposition 3.1.2.4(3) tells us that the image Y is complete and closed in V. But V is an affine variety, and so a closed subvariety is also affine. The only complete affine varieties are finite by Proposition 3.1.2.4(6), but since G/B is complete, this implies that Y is a single point, namely v. But then, for any $g \in G$, $\rho(g)L = L$, and so $g \in M$ by (3.7), whence $G = M$. Now M is a subgroup of N, and since Lemma 3.1.7.1 tells us that $[N : B]$ is finite, this tells us that $[M : B] = [G : B]$ is also finite. Proposition 1.3.1.1(5) tells us that every closed subgroup of G of finite index contains G°. But G is connected, so $G = G^\circ \subset B$, which implies $G = B$. But this cannot hold, since G is assumed to be semisimple. Therefore $S \neq \{e\}$ and we are done.

Suppose finally that G is not semisimple, so $R(G) \neq \{e\}$. Then $\dim G > \dim G/R(G)$ and so the induction hypothesis applies to the semisimple group $G/R(G)$. By Corollary 3.1.4.7, a Borel subgroup of $G/R(G)$ is of the form $B/R(G)$ for some Borel subgroup $B \subset G$. Moreover, we have just observed that this is equal to $N_G(B)/R(G)$. Take $x \in N_G(B)$. Then $xR(G) \in B/R(G)$, and so $xy_1 = by_2$ for $b \in B$ and $y_1, y_2 \in R(G)$. But it is a fact that $R(G) \subset B$, by Lemma 3.1.7.2. So $x = y_1^{-1}by_2 \in B$, and so $N_G(B) = B$, as required. □

Corollary 3.1.7.4. *Let P be a parabolic subgroup of G. Then $P = N_G(P)$. In particular, P is connected.*

Proof. We need to show that $N_G(P) \subset P$, so let $x \in N_G(P)$. By Corollary 3.1.4.6 we know that there is a Borel subgroup B of G such that $B \subset P$. Of course, since B is connected, we have $B \subset P^\circ \subset P$. Now $xBx^{-1} \subset P^\circ$, and is a Borel subgroup. But then both B and xBx^{-1} are Borel subgroups of P°, and so they are conjugate, that is, there exists an element $y \in P^\circ$ such that $yBy^{-1} = xBx^{-1}$. This in turn tells us $xyB(xy)^{-1} = B$, or $xy \in N_G(B)$. But Theorem 3.1.7.3 tells us $N_G(B) = B$, and so $xy \in B \subset P^\circ$. Now $xy, y \in P^\circ$, and so $x \in P^\circ \subset P$, whence $N_G(P) = P^\circ = P$, as required. □

Corollary 3.1.7.5. *Let P, Q be parabolic subgroups of G, both of which include a Borel subgroup B. If P, Q are conjugate in G, then $P = Q$.*

Proof. We have $B \subset P, B \subset Q$ and $x^{-1}Px = Q$ for some $x \in G$. Combining these, we see that $x^{-1}Bx \subset x^{-1}Px = Q$, and so B, $x^{-1}Bx$ are both Borel subgroups of Q. Now Q is connected by Corollary 3.1.7.4, and so we can apply Theorem 3.1.4.2. This means there is an element $y \in Q$ such that $y^{-1}x^{-1}Bxy = B$, or $xy \in N_G(B)$. But Theorem 3.1.7.3 then tells us $xy \in B \subset Q$, and this combined with $y \in Q$ tells us that $x \in Q$. But then $P = xQx^{-1} = Q$, as required. □

3.1.8 The Collection of Borel Subgroups

Proposition 3.1.8.1. *Let $\mathscr{B}$ be the collection of Borel subgroups of G, with B some fixed Borel subgroup. Then there is a map*

$$\alpha : \mathscr{B} \longrightarrow G/B$$

which is bijective.

Proof. First we describe the map α. Given an element $B' \in \mathscr{B}$, since B' and B are conjugate, there exists an element $x \in G$ such that $B' = xBx^{-1}$. We then define $\alpha(B') = xB$. Suppose, however, that $B' = yBy^{-1} = xBx^{-1}$. In order to show that α is well defined, we need to show that $xB = yB$. If $yBy^{-1} = xBx^{-1}$, then

$$B = y^{-1}yBy^{-1}y = y^{-1}(xBx^{-1}y^{-1} = (y^{-1}x)B(y^{-1}x)^{-1}$$

and so $y^{-1}x \in N_G(B) = B$, the latter equality being Theorem 3.1.7.3. But this says $xB = yB$ as required, and so the map is well defined.

The map α is clearly surjective, since for any $xB \in G/B$, the Borel subgroup xBx^{-1} maps to xB under α. Injectivity, too, is simple, for if $B', B'' \in \mathscr{B}$ are such that $\alpha(B') = \alpha(B'') = xB$, then $B' = xBx^{-1} = B''$, and so the map is injective. □

Recall, from Theorem 3.1.4.2, that G/B ($= \mathscr{B}$) is a projective variety. Moreover, there is a natural action of G on $\mathscr{B}$ by

$$B' \longmapsto xB'x^{-1}.$$

This action then goes over into the natural action of G on G/B

$$yB \longmapsto xyB.$$

That is to say, the following diagram, where the horizontal maps describe the respective G-actions, is a commutative diagram:

$$\begin{tikzcd} G \times \mathscr{B} \arrow[r] \arrow[d, "\mathrm{id}\times\alpha"] & \mathscr{B} \arrow[d, "\alpha"] \\ G \times G/B \arrow[r] & G/B \end{tikzcd}$$

Proposition 3.1.8.2. *For any subgroup H of G, the set of all Borel groups containing H (if any) corresponds to the set of fixed points of H on G/B, and is therefore closed.*

Proof. The set of fixed points of H on G/B is defined to be

$$(G/B)^H = \{xB \in G/B \mid hxB = xB, \forall h \in H\}$$

Suppose $xB \in (G/B)^H$. Then, for each $h \in H$, we have $hxB = xB$, or $h \in xBx^{-1}$, whence $H \subset xBx^{-1}$. So each element $xB \in (G/B)^H$ corresponds to a Borel subgroup xBx^{-1} which contains H. This correspondence is bijective, since it is given by the map α described in Proposition 3.1.8.1. □

Definition 3.1.8.3. We denote the first set described in Proposition 3.1.8.2 by $\mathscr{B}^H = \{B' \in \mathscr{B} \mid H \subset B'\}$, which we just saw is equal to $(G/B)^H$. The set $\mathscr{B}^H$ is intrinsically defined, while the set $(G/B)^H$ depends on the choice of B.

3.2 Root Systems

The key technique for understanding the structure of a reductive group is to decompose it via its root system. In this section we begin the task of understanding the ways in which an abstract root system is built in to a reductive group.

3.2.1 Regular Tori

Definition 3.2.1.1. If $\mathrm{Card}(\mathscr{B}^S)$ is finite, we call S *regular.* Otherwise, we say that S is *singular.*

The terminology here is taken from [6], although we note that, in [2], the term "semiregular torus" is used in place of "singular torus".

Proposition 3.2.1.2. *Let S be a torus in G, with $C = C_G(S)$. Then S is regular if and only if C is solvable. In that case, C lies in every Borel subgroup of G containing S, and for each maximal torus T such that $S \subset T$, we have*

$$\mathscr{B}^T = \mathscr{B}^S$$

Proof. Let $X = (G/B)^S$, a closed subset of G/B which we identify with $\mathscr{B}^S$. By Theorem 3.1.6.7, we know that C acts transitively on each connected component Y of X. In particular, for a fixed Borel subgroup B containing S, the map

$$\begin{aligned} C &\longrightarrow Y \\ x &\longmapsto xB \end{aligned}$$

is a surjective algebraic group homomorphism. Moreover, the kernel of this map is $C_B(S) = C \cap B$, and so $Y = C/C_B(S)$. As such, we have $\dim Y = \dim C - \dim C_B(S)$. Now, S is regular if and only if $\mathscr{B}^S$ is finite, which is the case if and only if $\dim X$ is zero. But $\dim X = \dim Y$, since Y is a connected component of X, and so S is regular if and only if $\dim Y = \dim C - \dim C_B(S) = 0$, which implies $C = C_B(S)$. But we know that $C_B(S)$ is a Borel subgroup of C, by Corollary 3.1.4.7, which means that S is regular if and only if C is solvable. Moreover, if this is the case, since $C = C_B(S)$ we have $C \subset B$. Finally, if T is a maximal torus of G which contains S, then clearly $T \subset C$. Therefore, if S is regular, then, for any $B \in \mathscr{B}^S$, we have $T \subset C \subset B$, and therefore $B \subset \mathscr{B}^T$. On the other hand, clearly any $B \in \mathscr{B}^T$ lies in $\mathscr{B}^S$, and therefore $\mathscr{B}^T = \mathscr{B}^S$ when S is regular. $\square$

Corollary 3.2.1.3. *All maximal tori are regular.*

Proof. Let T be a maximal torus of G. By Corollary 3.1.5.5, $C_G(T)^\circ = C_G(T)$ is nilpotent, and therefore solvable. Proposition 3.2.1.2 then says T is regular. $\square$

3.2.2 Weyl Groups

Definition 3.2.2.1. Let S be any torus in G. Then define a group

$$W(G,S) = N_G(S)/C_G(S)$$

We call this the *Weyl group of G relative to S.*

Proposition 3.2.2.2. *The group $W(G,S)$ is finite.*

Proof. Since S is a torus we have $C_G(S) = C_G(S)^\circ$. Furthermore, by Corollary 2.1.6.3, $N_G(S)^\circ = C_G(S)^\circ$, and so

$$[N_G(S) : C_G(S)^\circ] = [N_G(S) : N_G(S)^\circ]$$

and the latter is certainly finite. $\square$

Proposition 3.2.2.3. *All the Weyl groups of maximal tori are isomorphic.*

Proof. We know that all maximal tori are conjugate in G by Corollary 3.1.4.3. Therefore, given maximal tori $T, T' \subset G$, there exists an element $x \in G$ such that $T' = xTx^{-1}$. Let $N = N_G(T), N' = N_G(T')$ and $C = C_G(T), C' = C_G(T')$. We define a map

$$\begin{aligned} \psi : N/C &\longrightarrow N'/C' \\ gC &\longmapsto xgx^{-1}C' \end{aligned}$$

This map is well defined, since if $gC = hC$, then $h^{-1}g \in C$. We wish to show that

$$(xhx^{-1})^{-1}(xgx^{-1}) = (xh^{-1}x^{-1})xgx^{-1} = xh^{-1}gx^{-1}$$

lies in C'. Therefore, we consider

$$xh^{-1}gx^{-1}xTx^{-1}(xh^{-1}gx^{-1})^{-1} = xh^{-1}gThg^{-1}x^{-1} = xTx^{-1}$$

where the last equality is because $h^{-1}g \in C$. From this we conclude that $xh^{-1}gx^{-1} \in C'$ and so $xhx^{-1}C' = xgx^{-1}C'$ and the map ψ is well defined.

We note also that ψ is clearly a morphism of varieties, being simply multiplication by x and x^{-1}. Moreover, ψ is a group homomorphism, since, given $gC, hC \in N/C$, we have

$$\begin{aligned} \psi((gC)(hC)) = \psi(ghC) &= xghx^{-1}C' \\ &= xgx^{-1}xhx^{-1}C' = (xgx^{-1}C')(xhx^{-1})C' \end{aligned}$$

It remains to show that ψ is a group isomorphism. First we show injectivity, so suppose $gC \in \ker\psi$, that is, $xgx^{-1} \in C'$. Then

$$gTg^{-1} = x^{-1}(xgx^{-1}(xTx^{-1})(xgx^{-1})^{-1}x = x^{-1}(xTx^{-1})x = T$$

and so $g \in C$, whence ψ is injective. In order to show surjectivity, let $hC' \in N'/C'$. We then have $x^{-1}hx \in N$, since

$$x^{-1}hxT(x^{-1}hx)^{-1} = x^{-1}h(xTx^{-1})h^{-1}x = x^{-1}(xTx^{-1})x = T$$

where the second last equality holds because $h \in N'$. Now it is immediate that

$$\psi(x^{-1}hxC) = hC'$$

and so ψ is surjective, and is therefore an isomorphism, whence $W(G,T) \cong W(G,T')$ as required. $\square$

Thanks to Proposition 3.2.2.3, we will refer to a Weyl group of G relative to any maximal torus simply as *the* Weyl group of G, and denote it simply by W.

Lemma 3.2.2.4. *Let T be a maximal torus of G, and $C = C_G(T)$. If B is a Borel subgroup of G such that $T \subset B$, then $C \subset B$.*

Proof. Since, by Corollaries 3.1.5.5 and 3.1.6.6, we know that C is both connected and nilpotent, and therefore solvable, we know that it must be included in some Borel subgroup B'. But $T \subset C$, and therefore T is included in B' also. Now, there is some $x \in G$ such that $B' = xBx^{-1}$. Since $T \subset B$, we have $xTx^{-1} \in B'$. Therefore, both T and xTx^{-1} are maximal tori of B', and therefore are conjugate in B'. That is, there exists an element $y \in B'$ such that $yTy^{-1} = xTx^{-1}$. But then we have

$$xCx^{-1} = C_G(xTx^{-1}) = C_G(yTy^{-1}) = yCy^{-1}$$

or $C = x^{-1}yCy^{-1}x$. But then

$$C = x^{-1}yCy^{-1}x \subset x^{-1}yB'y^{-1}x = x^{-1}B'x = B$$

where the second last equality holds because $y \in B'$. $\square$

Proposition 3.2.2.5. *The action of G on $\mathscr{B}$ induces an action of the Weyl group W on the set $\mathscr{B}^T$,*

$$\begin{aligned} W \times \mathscr{B}^T &\longrightarrow \mathscr{B}^T \\ (xC, B) &\longrightarrow xBx^{-1} \end{aligned}$$

Proof. First we note that Lemma 3.2.2.4 implies that $C = C_G(T)$ acts trivially on $\mathscr{B}^T$. On the other hand, the action of $N = N_G(T)$ obviously stabilises $\mathscr{B}^T$, that is, $N.\mathscr{B}^T \subset \mathscr{B}^T$, and so we obtain an action of $W = N/C$ on $\mathscr{B}^T$. $\square$

Recall that to say a group H acts on a set Z *simply transitively* means first that the action is transitive, and secondly that the action is *free*, meaning that if $h.z = z$, then $h = e$.

Proposition 3.2.2.6. *Let T be a maximal torus of G, with $W = W(G,T)$. Then W acts on the set $\mathscr{B}^T$ simply transitively. In particular,* $\mathrm{Card}(\mathscr{B}^T) = |W|$ *is finite.*

Proof. We first show that the W-action is transitive. Let $B_1, B_2 \in \mathscr{B}^T$. Since Borel subgroups are conjugate, there exists an element $x \in G$ such that $B_1 = xB_2x^{-1}$. Since $T \subset B_2$, we have $xTx^{-1} \subset xB_2x^{-1} = B_1$, and so both T and xTx^{-1} are maximal tori of B_1. Hence there is an element $y \in B_1$ such that $yxTx^{-1}y^{-1} = T$, and therefore $yx \in N_G(T)$. Furthermore,

$$yxB_2x^{-1}y^{-1} = yB_1y^{-1} = B_1$$

where the latter equality holds since $y \in B_1$. This shows, then, that

$$yxC.B_2 = yxB_2x^{-1}y^{-1} = B_1$$

and so the action is transitive.

We now need to show that the action is free, so suppose that we have $B \in \mathscr{B}^T$ and $x \in N_G(T)$ such that $x.B = xBx^{-1} = B$. We need to show that $x \in C_G(T)$. We know that $B = N_G(B)$, so $x \in B \cap N_G(T) = N_B(T)$. But we know by Proposition 2.3.4.7 that $N_B(T) = C_B(T)$ since B is solvable and T consists of semisimple elements, and so $x \in C_B(T) \subset C_G(T)$, as required. $\square$

We also wish to define another W-action, this time on the character group of the maximal torus, $\mathrm{X}(T)$, as defined in Definition 2.1.3.2.

Lemma 3.2.2.7. *The group $N_G(T)$ acts on $\mathrm{X}(T)$ as follows: given $n \in N_G(T)$, and $\chi \in \mathrm{X}(T)$, define $n\chi \in \mathrm{X}(T)$ by*

$$(n\chi)(t) = \chi(n^{-1}tn)$$

for each $t \in T$.

Proof. Certainly it is true that $n\chi$ is a character of T, since it is a composite of group homomorphisms. Obviously $e\chi = \chi$, and if $n, m \in N_G(T)$, then

$$((nm)\chi)(t) = \chi(m^{-1}n^{-1}tnm) = m\chi(n^{-1}tn) = n(m\chi)(t)$$

and so this is a group action. $\square$

Proposition 3.2.2.8. *The group W acts on the character group $\mathrm{X}(T)$ as follows:*

$$\begin{aligned} W \times \mathrm{X}(T) &\longrightarrow \mathrm{X}(T) \\ (\sigma, \chi) &\longmapsto n\chi \end{aligned}$$

where $n \in N_G(T)$ is a representative for $\sigma \in W$.

Proof. Write $N = N_G(T), C = C_G(T)$. We first we need to show that the map is well defined, so let $n, m \in N_G(T)$ both be representatives for σ. That is, $nC = mC \in N/C$, we have $m^{-1}n \in C$, and so

$$(m^{-1}n)\chi(t) = \chi((m^{-1}n)^{-1}t(m^{-1}n)) = \chi(t(m^{-1}n)^{-1}(m^{-1}n)) = \chi(t)$$

Therefore $n\chi = m\chi$, and the map is well defined. Now, obviously if $e \in W$ is the identity, $e = eC$, and so $e\chi = \chi$. If $\sigma, \tau \in W$ are represented by $n, m \in N$ respectively, then $\sigma\tau$ is represented by nm, and so

$$(\sigma\tau)(\chi) = (nm)(\chi) = n(m\chi) = \sigma(\tau\chi)$$

where the middle equality holds because N acts of $\mathrm{X}(T)$. This proves that W acts on $\mathrm{X}(T)$. $\square$

Proposition 3.2.2.9. *Let $\varphi : G \to G'$ be an epimorphism of algebraic groups with $T \subset G$ a maximal torus. Then $T' = \varphi(T)$ is a maximal torus of G', and φ induces surjective maps*

$$\mathscr{B}^T \longrightarrow \mathscr{B}'^{T'} \tag{3.8}$$

and

$$W(G,T) \longrightarrow W(G',T') \tag{3.9}$$

If, in addition, $\ker\varphi$ lies in all Borel subgroups of G, then these maps are also injective.

Proof. We know that T' is a maximal torus by Corollary 3.1.4.7. This same result also told us that, given a Borel subgroup $B \subset G$, the image $\varphi(B)$ is a Borel subgroup of G'. Certainly, if $T \subset B$, then $\varphi(T) = T' \subset \varphi(B) = B'$, and so the map (3.8) is defined by sending $B \in \mathscr{B}^T$ to $\varphi(B) \in \mathscr{B}'^{T'}$.

Moreover, if $N = N_G(T), N' = N_{G'}(T')$, then, for $x \in N$, we have

$$\varphi(x)\varphi(T) = \varphi(xT) = \varphi(Tx) = \varphi(T)\varphi(x)$$

and so $\varphi(x) \in N'$, whence $\varphi(N) \subset N'$. Setting $C = C_G(T), C' = C_{G'}(T')$, if $x \in C, g \in T$, then

$$\varphi(x)\varphi(g) = \varphi(xg) = \varphi(gx) = \varphi(g)\varphi(x)$$

and so $\varphi(x) \in C'$, whence $\varphi(C) \subset C'$. Therefore the map (3.9) is also defined, sending $xC \in W$ to $\varphi(x)C' \in W' = W(G',T')$. Indeed, the map (3.9) is a clearly group homomorphism, since φ is.

Consider the map (3.8). Suppose $B' \in \mathscr{B}^{T'}$. We know that the map $\mathscr{B} \to \mathscr{B}'$ induced by φ is surjective by an Corollary 3.1.4.7, and so we can define a nonempty group $H = \varphi^{-1}(B')^{\circ}$. We note here that a Borel subgroup B of G which lies in H is certainly a Borel subroup of H. On the other hand, suppose $B_1 \subset H$ is a Borel subgroup of H, and suppose B_2 is a Borel subgroup of G such that $B_1 \subset B_2$. Now $\varphi(B_2)$ is a Borel subgroup of G', and

$$B' = \varphi(B_1) \subset \varphi(B_2)$$

whence $B' = \varphi(B_2)$. But then $B_2 \subset \varphi^{-1}(B')$, and, being connected, in fact lies in H. Therefore $B_1 = B_2$, and so the Borel subgroups of H are also Borel subgroups of G, and they each map to B'. Now certainly $T \subset H$, and so is a maximal torus of H. It is therefore contained in some Borel subgroup B of H.

But, as we noted above, if B is a Borel subgroup of H, then it is also a Borel subgroup of G and so $B \in \mathscr{B}^T$ and $\varphi(B) = B'$. Suppose, finally, that $\ker \varphi$ lies in all Borel subgroups of G, and that we have elements $B_1, B_2 \in \mathscr{B}$, such that $\varphi(B_1) = \varphi(B_2)$. Let $x \in B_1$. If we show that $x \in B_2$, then, by symmetry, we will have shown that $B_1 = B_2$, and therefore that the map $\mathscr{B} \to \mathscr{B}'$ induced by φ is injective. Since $\varphi(x) \in \varphi(B_1) = \varphi(B_2)$, there is an element $y \in B_2$ such that $\varphi(x) = \varphi(y)$, and so $\varphi(xy^{-1}) \in \ker \varphi \subset B_1 \cap B_2$. Then $xy^{-1}, y \in B_2$, and so, by closure, $x \in B_2$ as required. Since the map the map $\mathscr{B} \to \mathscr{B}'$ induced by φ is injective, it follows that the map (3.8) is injective.

We now turn to the map (3.9). Choose $B \in \mathscr{B}^T$, and define $B' = \varphi(B) \in \mathscr{B}'^{T'}$. Then the following diagram is commutative:

$$\begin{array}{ccc} W & \longrightarrow & W' \\ \downarrow & & \downarrow \\ \mathscr{B}^T & \longrightarrow & \mathscr{B}'^{T'} \end{array}$$

where the vertical maps are the orbit maps of B and B', respectively, and the horizontal maps are (3.8) and (3.9). Proposition 3.2.2.6 tells us that both the vertical maps are bijections. Moreover, we have shown above that the bottom horizontal map is surjective, and is bijective if $\ker \varphi$ lies in every Borel subgroup of G. Therefore the top horizontal map is itself surjective, and is bijective if $\ker \varphi$ lies in every Borel subgroup of G, as required. $\square$

3.2.3 The Roots of G

Let T be a torus, and let $\rho : T \to GL(V)$ be a rational representation. Since T is diagonalisable, we can decompose V as follows:

$$V = \bigoplus_{\chi \in \mathrm{X}(T)} V_\chi$$

where V_χ is the weight space associated to χ, that is

$$V_\chi = \{v \in V \mid \rho(g)v = \chi(g)v, \forall g \in G\}$$

In particular, since $\mathrm{Ad}T$ stabilises $\mathfrak{g}$, we can decompose $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathrm{X}(T)} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{\mathrm{x} \in \mathfrak{g} \mid \mathrm{Ad}t(\mathrm{x}) = \alpha(t)\mathrm{x}, t \in T\}$$

Definition 3.2.3.1. Let $\alpha \in \mathrm{X}(T)$ be nontrivial. Then α is a *root* of G relative to T if the subspace $\mathfrak{g}_\alpha$ is nonzero. Denote the set of roots of G relative to T by $\Phi(G, T)$.

If the context is clear, we write Φ for $\Phi(G,T)$. With this notion in hand we can refine the above decomposition of $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \mathfrak{c}_{\mathfrak{g}}(T) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \tag{3.10}$$

Lemma 3.2.3.2. $\mathrm{Card}(\Phi) \le \dim(G) - \dim(\mathfrak{c}_{\mathfrak{g}}(T))$.

Proof. This is a simple consequence of the decomposition in (3.10). Indeed, since each root space $\mathfrak{g}_\alpha$ is nonempty by definition, it is of dimension at least one. Therefore

$$\dim G = \dim(\mathfrak{c}_{\mathfrak{g}}(T)) + \sum_{\alpha\in\Phi} \dim(\mathfrak{g}_\alpha) \ge \dim(\mathfrak{c}_{\mathfrak{g}}(T)) + \sum_{\alpha\in\Phi} 1$$

but this final term is equal to $\mathrm{Card}(\Phi)$, and we get the desired inequality. $\square$

Proposition 3.2.3.3. *Let $\sigma \in W = W(G,T)$ be represented by $n \in N_G(T)$. Then, for any root $\alpha \in \Phi$,*

$$\mathfrak{g}_{\sigma(\alpha)} = \mathrm{Ad}n(\mathfrak{g}_\alpha)$$

and so is nonempty. Therefore, the action of W on characters induces an action of W on the set of roots Φ, and, indeed, an action on root spaces, given by

$$\sigma.(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$$

Proof. First we recall from Lemma 3.2.2.7 that $\sigma\alpha(t) = \alpha(n^{-1}tn)$ for all $t \in T$. Suppose $\alpha \in \Phi$, and let $\mathrm{x} \in \mathrm{Ad}n(\mathfrak{g}_\alpha)$, so $\mathrm{x} = n\mathrm{x}'n^{-1}$ for some $\mathrm{x}' \in \mathfrak{g}_\alpha$. Suppose further that $t \in T$, and write $t' = n^{-1}tn$, so $t = nt'n^{-1}$. Now

$$\begin{aligned}\mathrm{Ad}(t)(\mathrm{x}) = t(n\mathrm{x}'n^{-1})t^{-1} &= n(n^{-1}tn)\mathrm{x}'(n^{-1}t^{-1}n)n^{-1} \\ &= \alpha(n^{-1}tn)n\mathrm{x}'n^{-1} = (\sigma\alpha)(t)\mathrm{x}\end{aligned}$$

and so $\mathrm{x} \in \mathfrak{g}_{\sigma\alpha}$, therefore $\mathrm{Ad}n(\mathfrak{g}_\alpha) \subset \mathfrak{g}_{\sigma(\alpha)}$. But this also gives

$$\mathrm{Ad}(n^{-1})\mathfrak{g}_{\sigma\alpha} = (\mathrm{Ad}n)^{-1}\mathfrak{g}_{\sigma\alpha} \subset \mathfrak{g}_\alpha$$

in the case $\sigma\alpha$ happens to be a root, and is trivially true otherwise. It follows, therefore, that $\mathfrak{g}_{\sigma\alpha} \subset \mathrm{Ad}n\mathfrak{g}_\alpha$, as required. The fact that this induces a W-action on Φ and also on the collection of root spaces is now clear. $\square$

3.2.4 The Action of the Torus on Projective Space

Throughout this section, we will regard elements of $\mathbb{P}^1$ as elements of $\mathbb{G}_m \cup \{0,\infty\}$, by taking the correspondence $\begin{pmatrix} a \\ b \end{pmatrix} \leftrightarrow a/b$. In particular,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow \infty$$

It may also be helpful to view the projective variety $\mathbb{P}(V)$ as the quotient G/P, where $G = GL(n, K)$ and

$$P = \begin{pmatrix} GL(n-1, K) & 0 \\ * & \mathbb{G}_m \end{pmatrix}$$

Indeed, with this set up, $P = \{x \in GL(n, K) \,|\, xL = L\}$, where L is the 1-dimensional subspace of V spanned by the vector

$$e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

As we saw in §1.5.2, the homogeneous space G/P is then identified with the orbit of the natural G-action on $[L] \in \mathbb{P}(V)$. But given any element $[v] \in \mathbb{P}(V)$, if we set $x = [x'|v]$, where x' is an $(n-1) \times n$ matrix, then $[xL] = [v]$. Therefore, the orbit of the G-action on $[L]$ is all of $\mathbb{P}(V)$, or $G/P = \mathbb{P}(V)$.

Definition 3.2.4.1. Let T be a torus. We call a morphism of algebraic groups $\lambda : \mathbb{G}_m \to T$ a *1 parameter subgroup.* We denote the set of 1 parameter subgroups of T by $\mathrm{Y}(T)$.

There is a natural pairing between $\mathrm{X}(T)$ and $\mathrm{Y}(T)$: note that $\lambda \circ \chi : \mathbb{G}_m \to \mathbb{G}_m$ is a morphism of algebraic groups, and so is an element of $\mathrm{X}(\mathbb{G}_m)$. We saw in Proposition 2.1.3.5 that $\mathrm{X}(\mathbb{G}_m) = \mathbb{Z}$ (indeed, we saw that $\mathrm{X}(T) = \mathbb{Z}^l$, where l is the rank of the torus T), so each such composition of elements from $\mathrm{X}(T)$ and $\mathrm{Y}(T)$ corresponds to an integer.

Definition 3.2.4.2. Given elements $\chi \in \mathrm{X}(T)$ and $\lambda \in \mathrm{Y}(T)$, define an integer $\langle \chi, \lambda \rangle$ such that $\lambda \circ \chi(x) = x^{\langle \chi, \lambda \rangle}$ for all $x \in \mathbb{G}_m$.

It is evident that $\mathrm{X}(T)$ and $\mathrm{Y}(T)$ are dual $\mathbb{Z}$-modules, since $\mathrm{X}(T) = \mathbb{Z}^l$ and so $\mathrm{Y}(T) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}(T), \mathbb{Z}) = \mathbb{Z}^l$. We will also need the following lemma later:

Lemma 3.2.4.3. *The Weyl group W of G acts on $\mathrm{Y}(T)$ as follows: for $\lambda \in \mathrm{Y}(T)$ and $\sigma \in W$ represented by $n \in N_G(T)$, define*

$$\sigma(\lambda)(a) = n\lambda(a)n^{-1}$$

for all $a \in \mathbb{G}_m$. Moreover, for any $\chi \in \mathrm{X}(T)$, $\lambda \in \mathrm{Y}(T)$ and $\sigma \in W$,

$$\langle \sigma\chi, \sigma\lambda \rangle = \langle \chi, \lambda \rangle$$

Proof. Certainly it is true that $\sigma\lambda$ is a 1 parameter subgroup of T, since it is a composite of the group homomorphisms λ followed by $\mathrm{Int} n$, where the latter stabilises T since $n \in N_G(T)$. $e\lambda = \lambda$, and if $\sigma, \tau \in W$ are represented by $n, m \in N_G(T)$, respectively then

$$(\sigma\tau)\lambda(a) = nm\lambda(a)m^{-1}n^{-1} = n(\tau\lambda)(a)n^{-1} = \sigma(\tau(lambda))(a)$$

and so this is a group action.

Recall from Lemma 3.2.2.7 that, for $\chi \in \mathrm{X}(T)$ and $\sigma \in W$ represented by $n \in N_G(T)$, we have

$$\sigma\chi(t) = \chi(n^{-1}tn)$$

for all $t \in T$. Suppose now that $\langle \chi, \lambda \rangle = m$, so $\chi \circ \lambda(a) = a^m$ for all $a \in \mathbb{G}_m$. Let $\sigma \in W$ be represented by $n \in N_G(T)$. Then, for $a \in \mathbb{G}_m$,

$$(\sigma\chi) \circ (\sigma\lambda)(a) = (\sigma\chi)(n\lambda(a)n^{-1}) = \chi(n^{-1}(n\lambda(a)n^{-1})n) = \chi(\lambda(a))$$

and so $\langle \sigma\chi, \sigma\lambda \rangle = \langle \chi, \lambda \rangle$, as required. □

Proposition 3.2.4.4. *Let $T \subset GL(V)$ be a torus and $\lambda : \mathbb{G}_m \to T$ a 1 parameter subgroup of T, so that $\mathbb{G}_m$ acts on $\mathbb{P}(V)$ via $t.[v] = [\lambda(t)v]$. If $v \in V$, then*

1. *The orbit map of the $\mathbb{G}_m$-action on $[v]$ can be extended to a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}(V)$.*
2. *The points $\varphi(0), \varphi(\infty) \in \mathbb{P}(V)$ are both fixed points of the $\lambda(\mathbb{G}_m)$-action on $\mathbb{P}(V)$.*
3. *$[v]$ is fixed by $\lambda(\mathbb{G}_m)$ if and only if $\varphi(0) = \varphi(\infty)$*
4. *λ can be chosen such that the fixed points of the T-action on $\mathbb{P}(V)$ coincide with the fixed points of the $\lambda(\mathbb{G}_m)$-action on $\mathbb{P}(V)$.*

Proof. Proposition 2.1.5.6 says that, since T is diagonalisable, we can find a basis of V consisting of eigenvalues, $\{v_1, \dots, v_n\}$. Each of these vectors determines a character $\gamma_i \in \mathrm{X}(T)$, such that $v_i \in V_{\gamma_i}$. Let $m_i = \langle \gamma_i, \lambda \rangle$, then if $v = \sum a_i v_i$,

$$\lambda(t)v = \lambda(t) \sum a_i v_i = \sum a_i \lambda(t) v_i = \sum a_i \gamma_i(\lambda(t)) v_i = \sum a_i t^{m_i} v_i$$

for all $t \in \mathbb{G}_m$.

Let $v = \sum a_i v_i$, and set $I = \{i \,|\, a_i \neq 0\}$. Introducing more notation, we define

$$m_0 = \min_{i \in I}\{m_i\}, \qquad I_0 = \{i \in I \,|\, m_i = m_0\}$$
$$m^0 = \max_{i \in I}\{m_i\}, \qquad I^0 = \{i \in I \,|\, m_i = m^0\}$$

Finally, we define maps

$$\varphi_0(t) = \left[\sum a_i t^{m_i - m_0} v_i\right], \qquad t \in \mathbb{G}_m \cup \{0\}$$
$$\varphi^0(t) = \left[\sum a_i t^{m_i - m^0} v_i\right], \qquad t \in \mathbb{G}_m \cup \{\infty\}$$

These formulae make sense since $m_i - m_0 \geq 0$ for all i, so $t^{m_i - m_0}$ is well defined for all $t \in \mathbb{G}_m \cup \{0\}$, and since $m_i - m^0 \leq 0$ for all i, so $t^{m_i - m^0} = (1/t)^{m^0 - m_i}$

is well defined for all $t \in \mathbb{G}_m \cup \{\infty\}$. Furthermore, since $t^{-m_0}, t^{-m^0} \in K^*$ for all $t \in \mathbb{G}_m$, it follows that $[t^{-m_0}v] = [t^{-m^0}v] = [v]$, and so we have the equality

$$\varphi_0|_{\mathbb{G}_m} = \varphi = \varphi^0|_{\mathbb{G}_m}$$

where $\varphi : t \to [\lambda(t)v]$. That is, φ^0 and φ_0 are two morphisms which agree on an open affine set $\mathbb{G}_m$, and so can be glued together to give a morphism on $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$

$$\begin{aligned} \varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}(V) \\ t &\longmapsto [\lambda(t)v] \end{aligned}$$

where we have used the notation

$$\begin{aligned} [\lambda(0)v] &= \varphi_0(0) = \left[\sum_{i \in I_0} a_i v_i\right] \\ [\lambda(\infty)v] &= \varphi^0(\infty) = \left[\sum_{i \in I^0} a_i v_i\right] \end{aligned} \tag{3.11}$$

This proves (1).

We now need to consider the $\lambda(\mathbb{G}_m)$-action on $\varphi(0), \varphi(\infty) \in \mathbb{P}(V)$, so let $g = \lambda(t)$ for some $t \in \mathbb{G}_m$. Firstly,

$$\lambda(t).[\lambda(0)v] = \left[\sum_{i \in I_0} a_i t^{m_i} v_i\right] = \left[\sum_{i \in I_0} a_i t^{m_0} v_i\right] = [t^{m_0}\lambda(0)v] = [\lambda(0)v]$$

and, similarly,

$$\lambda(t).[\lambda(\infty)v] = \left[\sum_{i \in I^0} a_i t^{m_i} v_i\right] = \left[\sum_{i \in I^0} a_i t^{m^0} v_i\right] = [t^{m^0}\lambda(\infty)v] = [\lambda(\infty)v]$$

Therefore $\varphi(0)$ and $\varphi(\infty)$ are fixed under the $\lambda(\mathbb{G}_m)$-action, proving (2).

For (3), suppose firstly that $[v]$ is fixed by the $\lambda(\mathbb{G}_m)$-action. This means $[v] = [\lambda(t)v]$ for all $t \in \mathbb{G}_m$, that is

$$\left[\sum a_i v_i\right] = \left[\sum a_i t^{m_i} v_i\right]$$

which means there is some constant $k \in K^*$ such that $t^{m_i} = k$ for all i. But this means $m_0 = m^0$, and therefore $I_0 = I^0$. But plugging this last fact in to the formulae given in (3.11) implies

$$\varphi(0) = \varphi_0(0) = \varphi^0(\infty) = \varphi(\infty)$$

and our two fixed points coincide. On the other hand, suppose $\varphi(0) = \varphi(\infty)$. Then we see immediately that this implies $I_0 = I^0$, and so $m_0 = m^0$. But this then gives us $m_i = m_0 = m^0$ for all $i \in I$, and so, if $t \in \mathbb{G}_m$,

$$[\lambda(t)v] = \left[\sum_{i \in I} a_i t^{m_i} v_i\right] = \left[t^{m_0} \sum_{i \in I} a_i v_i\right] = [t^{m_0}v] = [v]$$

which means $[v]$ is fixed by the $\lambda(\mathbb{G}_m)$-action, thus proving (3).

We now turn to (4). Note firstly that, since $\mathrm{X}(T)$ and $\mathrm{Y}(T)$ are dual $\mathbb{Z}$-modules, we can always find $\lambda \in \mathrm{Y}(T)$ in such a way that if $\gamma_i \neq \gamma_j$ then $m_i = \langle \gamma_i, \lambda \rangle \neq \langle \gamma_j, \lambda \rangle = m_j$. Let λ be of this type.

Suppose further that $w = \sum b_i v_i$ is an eigenvector of the $\lambda(\mathbb{G}_m)$-action, and so, for $t \in \mathbb{G}_m$, there is a $k \in K^*$ such that

$$\sum k b_i v_i = kw = \lambda(t) w = \sum b_i t^{m_i} v_i$$

which implies $t^{m_i} = k$ for all i such that $b_i \neq 0$. But since the m_i's are assumed to be distinct, this forces all but one of the b_i's to be zero, which in turn means $w = b_j v_j$ for some j. This means $w \in V_{\gamma_j}$ and so w is an eigenvector of the T-action, too. Therefore, if the m_i's are distinct, the eigenvectors of the T-action coincide with the eigenvectors of the $\lambda(\mathbb{G}_m)$-action.

In particular, if $[v]$ is fixed by the T-action, then it is an eigenvalue of T, and so is an eigenvalue of $\lambda(\mathbb{G}_m)$. But this means that $[v]$ is fixed by the $\mathbb{G}_m$-action, which means that $\varphi(0) = \varphi(\infty)$. On the other hand, if $\varphi(0) = \varphi(\infty)$, then $[v]$ is fixed by the $\mathbb{G}_m$-action, which means that v is an eigenvalue of the $\mathbb{G}_m$-action. But this means that v is an eigenvalue of the T-action, and so $[v]$ is fixed by the T-action.

□

Lemma 3.2.4.5. *Suppose W is a hyperplane in a vector space V. Suppose further that X is a closed, irreducible subvariety of $\mathbb{P}(V)$, with $(\mathbb{P}(V) - \mathbb{P}(W)) \cap X \neq \emptyset$. If*

1. *If $\dim X > 0$, then $X \cap \mathbb{P}(W) \neq \emptyset$.*
2. *If Y is an irreducible component of $X \cap \mathbb{P}(W)$, then $\dim Y = \dim X - 1$.*

Proof. For (1), note firstly that X is complete, since it is a closed subvariety of the complete variety $\mathbb{P}(V)$. Secondly, if $(v_1, \ldots, v_n)$ is a basis for V, and $W = \mathrm{span}(v_2, \ldots, v_n)$, the map

$$\begin{aligned} \mathbb{P}(V) - \mathbb{P}(W) &\longrightarrow \mathbf{A}^{n-1} \\ \left[\sum a_i v_i\right] &\longmapsto (a_2/a_1, \ldots, a_n/a_1) \end{aligned}$$

is clearly an isomorphism of varieties. Therefore, if $X \subset \mathbb{P}(V) - \mathbb{P}(W)$, by Proposition 3.1.2.4(6), $\dim X = 0$, which is contrary to our assumption. Therefore, $X \cap \mathbb{P}(W) \neq \emptyset$.

For (2), let U_i be the open affine subset of $\mathbb{P}(V)$ consisting of vectors with nonzero i^{th} coordinate. Define a map

$$\begin{aligned} f : U_i &\longrightarrow K \\ \left[\sum a_i v_i\right] &\longmapsto a_1/a_i \end{aligned}$$

Then f lies in $K[U_i]$, and indeed $U_i \cap \mathbb{P}(W) = \mathscr{V}(f)$. Moreover, $U_i \cap \mathbb{P}(W) \cap Y = \mathscr{V}_Y(f)$. We now apply a result from algebraic geometry, given in A.G. 9.2 of [2], which says that if V is an irreducible variety, and $f \in K[V]$ is a non-constant function whose zero set $\mathscr{V}_V(f)$ in V is not empty, then the dimension of each irreducible component of $\mathscr{V}_V(f)$ is $\dim V - 1$. This applies readily to our situation, and tells us that each connected component of $\mathbb{P}(W) \cap Y$ has dimension $\dim Y - 1$. □

Proposition 3.2.4.6. *Let $T \subset GL(V)$, acting naturally on $\mathbb{P}(V)$. Suppose Y is a closed, irreducible subvariety of $\mathbb{P}(V)$ which is T-stable. Then*

1. *If $\dim Y \geq 1$, T fixes at least 2 points of Y.*
2. *If $\dim Y \geq 2$, T fixes at least 3 points of Y.*

Proof. As in Proposition 3.2.4.4, choose characters $\gamma_i \in \mathrm{X}(T)$ such that $\{v_1, \ldots, v_n\}$ is a basis of V with $v_i \in V_{\gamma_i}$. Furthermore, we also choose a 1 parameter subgroup $\lambda \in \mathrm{Y}(T)$ such that $m_i = \langle \gamma_i, \lambda \rangle$, with the property that $m_i \neq m_j$ if $\gamma_i \neq \gamma_j$. Proposition 3.2.4.4(4) now says that the eigenvalues of the T-action coincide with the eigenvalues of the $\lambda(\mathbb{G}_m)$-action, which means we only need to find fixed points of the $\lambda(\mathbb{G}_m)$-action in order to prove this proposition. Therefore, we may assume $\lambda(\mathbb{G}_m) = T$.

For (1), suppose $\dim Y \geq 1$. This then implies that Y has infinitely many points, which means that if T fixes all the points of Y, it certainly fixes more than 2. On the other hand, suppose T does not fix all of Y, and in particular, let $[v] \in Y$ be an element which is not fixed by the T-action.

By Proposition 3.2.4.4(2) applied to v, we see that $[\lambda(0)v], [\lambda(\infty)v] \in \mathbb{P}(V)$ are both fixed by the T-action, while Proposition 3.2.4.4(3) assure us that $[\lambda(0)v] \neq [\lambda(\infty)v]$ since $[v]$ is not fixed under the T-action. It remains to show that both of these are in fact elements of Y. We know that the map

$$\begin{aligned} f : \mathbb{P}^1 &\longrightarrow \mathbb{P}(V) \\ x &\longmapsto \lambda(x)[v] \end{aligned}$$

is a morphism of varieties. Since $\mathbb{P}^1$ is complete, Proposition 3.1.2.4(3) tells us that $f(\mathbb{P}^1)$ is closed. Now, certainly $f(\mathbb{P}^1) - \{f(0), f(\infty)\} \subset Y$, which means

$$Y \supset \overline{f(\mathbb{P}^1) - \{f(0), f(\infty)\}} = \overline{f(\mathbb{P}^1)} = f(\mathbb{P}^1)$$

as required.

For (2), suppose $\dim Y \geq 2$. Then Y is again infinite, and so if T fixes all points of Y then it certainly fixes three points, so we may assume that there is an element $[v] \in Y$ which is not fixed by T. Let $v = \sum a_i v_i$, and apply Proposition 3.2.4.4 to this vector. We can rearrange the labelling of the basis vectors if necessary to ensure that $a_1 \neq 0$ and $m_1 = m^0$. Let $W = \operatorname{span}\{v_2, \ldots, v_n\}$, so$v \notin W$, and therefore $[v] \notin \mathbb{P}(W)$. Then the conditions of Lemma 3.2.4.5 are met, and so we can take an irreducible component $Y' \subset Y \cap \mathbb{P}(W)$ with $\dim Y' = \dim Y - 1 \geq 1$. Note further that Y' is stable under the action of T, since Y and $\mathbb{P}(W)$ are. Because Y' is closed and irreducible, we can apply (1) to obtain two points $[y], [y'] \in Y' \subset$ which are fixed under the T-action.

Note now that, since $1 \in I^0$ and $a_1 \neq 0$, it follows that $[\lambda(\infty)v] = \left[\sum_{i \in I^0} a_i v_i\right]$ is not contained in $\mathbb{P}(W)$, and therefore is in not contained in Y', and so cannot be equal to $[y]$ or $[y']$. But Proposition 3.2.4.4 says $[\lambda(\infty)v]$ is fixed by the T-action, and we saw in the proof of (1) that $[\lambda(\infty)v] \in Y$ since $[v] \in Y$. We have therefore shown that the distinct points $[y], [y'], [\lambda(\infty)v] \in Y$ are fixed under the action of T. □

Proposition 3.2.4.7. *1. Let P be a proper parabolic subgroup of G. Then T fixes at least 2 distinct points of G/P and fixes at least 3 distinct points if $\dim G/P > 1$.*

2. If G is nonsolvable, a maximal torus of G lies in at least two Borel subgroups.

3. $W = \{e\}$ if and only if G is solvable.

4. If $|W| = 2$ then $\dim G/B = 1$ for any Borel subgroup $B \subset G$.

5. G is generated by the set $\mathscr{B}^T$, that is, the Borel subgroups containing T.

Proof. We start by proving (1). Use Theorem 1.5.1.2 to obtain a representation $\rho : G \to GL(V)$ and a line $L \subset V$ such that P is equal to the stabiliser of L under the G-action on V arising from ρ. This allows us to view G/P as the G-orbit of $[L] \subset \mathbb{P}(V)$. Denote this orbit by Y. We can furthermore assume us that we can in fact ensure G/P is embedded in $\mathbb{P}(V)$ such that it is not contained in any $\mathbb{P}(W)$ where W is a proper subspace of V, since otherwise we can replace V by $W = \operatorname{span}(\pi^{-1}(G/P))$, where π is the map sending nonzero vectors to their projective images. Since G/P is complete by definition, Y is the image of a complete variety, and so Proposition 3.1.2.4 (3) says Y is itself closed and complete. But since G is assumed to be connected, G/P is irreducible, and thus so is Y. Therefore the conditions of Proposition 3.2.4.6 are met, and so the T-action on Y fixes at least 2 points if $\dim Y = \dim G/P \geq 1$, or at least 3 if $\dim Y = \dim G/P > 1$.

For (2), we assume G is not solvable, and so necessarily contains a proper Borel subgroup B. We can therefore apply (1) to G/B to get at least 2 distinct fixed points of the T-action on G/B, namely xB and $x'B$. That is, for all $t \in T$, we have $txB = xB$, or $T \subset xBx^{-1}$. Similarly, $T \subset x'Bx'^{-1}$. Therefore T is contained in at least two Borel subgroups.

For (3), suppose firstly that G is not solvable. Then, by (2), T is contained in at least two Borel subgroups, and so $|W| = \operatorname{Card}(\mathscr{B}^T) \geq 2$. We conclude, therefore, G is solvable if $W = \{e\}$. On the other hand, if G is solvable, then Proposition 2.3.4.7 says $N_G(T) = C_G(T)$ and so W is trivial, as required.

We now turn to (4), so assume $|W| = 2$. Note firstly that Proposition 3.2.2.6 tells us $|W| = \operatorname{Card}(\mathscr{B}^T)$, and so T is contained in 2 distinct Borel subgroups. Let B be one of these; then G/B is infinite, so $\dim G/B \geq 1$. If $\dim G/B > 1$, then (1) tells us there are at least three fixed points of G/B under the T-action.

As we saw above, a fixed point of the T-action corresponds to a Borel subgroup containing T, and therefore $|W| = \mathrm{Card}(\mathscr{B}^T) \geq 3$, which is contrary to our assumption. Therefore, $\dim G/B = 1$.

Finally, for (5), we use induction on $\dim G$. If $\dim G = 0$, then $G = \{e\}$ since it is assumed connected, and so the statement is vacuously true. Consider, now, the closure of the group generated by $\mathscr{B}^T$, and call it H. Then, Corollary 3.1.4.6 shows that H is parabolic. Supposing $H \neq G$, we can apply (1) to get at least 2 fixed points of the action of T on G/H. In particular, there is an element $xH \in G/H$ distinct from H such that $txH = xH$. Write $H' = xHx^{-1}$, and note that $T \subset H'$ is a maximal torus. Moreover, since $N_G(H) = H$ by Corollary 3.1.7.4, $xHx^{-1} \neq H$.

Suppose now that B is a Borel subgroup of H'. Since H' is parabolic, by Corollary 3.1.4.6 there is a Borel subgroup B' of G which is contained in B'. By maximality of Borel subgroups, it is clear that B' is also a Borel subgroup of P, and so there is an element $x \in P$ such that $xBx^{-1} = B'$. This then shows that B is conjugate to a Borel subgroup of G, and so is itself a Borel subgroup of G.

Since $\dim H' < \dim G$, by induction H' is generated by its Borel subgroups containing T. But we just showed that Borel subgroups of H' which contain T are also Borel subgroups of G, and so $H' \subset H$. Since these two groups are conjugate, this then implies $H' = H$, which is a contradiction. Therefore, $H = G$, and the result is proven. □

3.3 Groups of Semisimple Rank 1

This section is something of a technical pit stop. Nevertheless, it should be familiar from the study of Lie Algebra just how useful it is to have a little knowledge of the small but abundant semisimple subalgebra $\mathfrak{sl}(2, K)$. Likewise, we frequently fall back upon subgroups of semisimple rank 2, which are somehow hidden in the reductive groups we wish to study.

3.3.1 The Groups Z_α

We begin this section by defining a group

$$I(T) = \Big(\bigcap_{B \in \mathscr{B}^T} B \Big)^\circ$$

Note that by Lemma 3.2.2.4, $C_G(T) \subset I(T)$. For each $\alpha \in \Phi$, we write

$$\mathfrak{g}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}'_\alpha$$

where

$$\mathfrak{h}_\alpha = \mathscr{L}(I(T)) \cap \mathfrak{g}_\alpha \tag{3.12}$$

Then we have a decomposition

$$\mathfrak{g} = \mathscr{L}(I(T)) \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}'_\alpha \tag{3.13}$$

where

$$\Psi = \{\alpha \in \Phi \mid \mathfrak{g}'_\alpha \neq 0\}$$

We say that the elements of Ψ are the roots which lie outside of $I(T)$. Note that one consequence of (3.12) is that, if $I(T) \subset C_G(T)$, then $\mathfrak{h}_\alpha = 0$, since otherwise $\mathfrak{g}_\alpha \cap \mathscr{L}(C_G(T)) \neq 0$, which is impossible. Therefore, we have the immediate lemma:

Lemma 3.3.1.1. *If $I(T) \subset C_G(T)$, then $\Psi = \Phi$.*

With this set up, we now define

$$T_\alpha = \ker \alpha^\circ \subset T$$

for $\alpha \in \Phi$. Now T_α is a torus of codimension 1 in T. We wish to show that T_α is singular.

Proposition 3.3.1.2. *Let S be a torus in G, and T a maximal torus such that $S \subset T$. Then S is singular if and only if $S \subset T_\alpha$ for some $\alpha \in \Psi$.*

Proof. First suppose that S is singular. Then, by Proposition 3.2.1.2, $C = C_G(S)$ is nonsolvable. However, since $C \cap I(T) \subset B$ for any $B \subset \mathscr{B}^T$, we have that $C \cap I(T)$ is solvable. Therefore C is of larger dimension than $C \cap I(T)$.

Suppose $\mathscr{L}(C) = \mathfrak{c}_\mathfrak{g}(S)$ is wholly contained in $\mathscr{L}(I(T))$. Therefore $\mathscr{L}(C) \cap \mathscr{L}(I(T)) = \mathscr{L}(C)$. On the other hand, since $C \cap I(T) \subset C$ we have $\mathscr{L}(C \cap I(T))\mathscr{L}(C)$ and similarly $C \cap I(T) \subset I(T)$ implies $\mathscr{L}(C \cap I(T)) \subset \mathscr{L}(I(T))$. Therefore $\mathscr{L}(C \cap I(T)) \subset \mathscr{L}(C) \cap \mathscr{L}(I(T)) = \mathscr{L}(C)$. But this implies $\dim(C \cap I(T)) \leq \dim C$ which is contrary to the first part of our proof.

We conclude, therefore, that $\mathscr{L}(C) = \mathfrak{c}_\mathfrak{g}(S)$ is not wholly contained in $\mathscr{L}(I(T))$, so suppose $\mathsf{x} \in \mathfrak{c}_\mathfrak{g}(S) - \mathscr{L}(I(T))$, that is, an element x which is fixed by S, but which does not lie in $\mathscr{L}(I(T))$, or, equivalently, which has a component in $\mathfrak{g}'_\alpha$ for some $\alpha \in \Psi$, call it x_α. Since S fixes x, it certainly fixes this component x_α, since $\mathfrak{g}'_\alpha$ is stable under $\mathrm{Ad}T$. But, for all $s \in S \subset T$, since $\mathsf{x}_\alpha \in \mathfrak{g}'_\alpha \subset \mathfrak{g}_\alpha$, we have $\mathrm{Ad}s(\mathsf{x}_\alpha) = \alpha(s)\mathsf{x}_\alpha$. Combining these two results, we see that $\alpha(s) = 1$ for all $s \in S$, that is, $S \subset \ker \alpha$. Since S is connected, this implies $S \subset T_\alpha$.

On the other hand, suppose S is regular. Then Proposition 3.2.1.2 again tells us that C is solvable, and, indeed, that $C \subset B$ for all $B \subset \mathscr{B}^T$. In particular, this tells us that $C \subset I(T)$. We know, by Proposition 2.3.1.9, that $\mathfrak{c}_\mathfrak{g}(S) = \mathscr{L}(C)$, and therefore

$$\mathfrak{c}_\mathfrak{g}(S) \subset \mathscr{L}(I(T)). \tag{3.14}$$

Let $\alpha \in \Psi$. Then, for any element $\mathsf{x} \in \mathfrak{g}'_\alpha$, we have

$$\mathrm{Ad}t(\mathsf{x}) = \alpha(t)\mathsf{x}, \quad \forall t \in T$$

Now, $\mathfrak{c}_\mathfrak{g}(S) \cap \mathfrak{g}'_\alpha = 0$ by equations (3.13) and (3.14). It follows that there is a nonzero $\mathsf{x} \notin \mathfrak{c}_\mathfrak{g}(S)$, and so there is an element $x \in S$ such that $\mathrm{Ad}x(\mathsf{x}) = \alpha(x)\mathsf{x} \neq \mathsf{x}$, that is, $\alpha(x) \neq 1$. But this means $x \notin \ker \alpha$, and in particular, $x \notin T_\alpha$, which in turn gives us $S \not\subseteq T_\alpha$. But since α was arbitrary, this tells us $S \not\subseteq T_\alpha$ for all $\alpha \in \Psi$, and the result is proven. $\square$

We now introduce a collection of very important subgroups of G.

Definition 3.3.1.3. In the notation of (3.13), for each $\alpha \in \Psi$, define

$$Z_\alpha = C_G(T_\alpha)$$

Example 29. Let $G = SL(n, K)$. Take

$$T = \{\mathrm{diag}(t_1, \ldots, t_n) \,|\, t_n = \frac{1}{t_1 \cdots t_{n-1}}\} \cong D(n-1, K)$$

We will see in Chapter 5 that $\alpha \in \Psi$ is of the form

$$\alpha : \mathrm{diag}(t_1, \ldots, t_n) \longmapsto t_i/t_j,\ i \neq j \tag{3.15}$$

and so we have

$$T_\alpha = \{\mathrm{diag}(t_1, \ldots, t_n) \in T \,|\, t_i = t_j\} \cong D(n-2, K)$$

Setting $Z_\alpha = C_G(T_\alpha)$ as above, we have $x \in Z_\alpha$ if and only if $txt^{-1} = x$ for all $t \in T_\alpha$, that is

$$(txt^{-1})_{rs} = (t_r/t_s)x_{rs} = x_{rs}$$

This forces $x_{rs} = 0$ unless

1. $r = s$, or
2. $r = i$ and $s = j$, or
3. $r = j$ and $s = i$.

That is, $x \in Z_\alpha$ has zero entries everywhere except on the diagonal, and in both the $(i, j)^{\text{th}}$ and $(j, i)^{\text{th}}$ positions.

Proposition 3.3.1.4. *$G_\alpha = Z_\alpha/R(Z_\alpha)$ is a semisimple group of rank 1.*

Proof. Note first that $T \subset Z_\alpha$, so T is a maximal torus of Z_α, and therefore the quotient image of T is a maximal torus of G_α. Of course T_α is singular by Proposition 3.3.1.2, and so Proposition 3.2.1.2 tells us that Z_α is non-solvable. Amongst other things, this tells us that $Z_\alpha \neq R(Z_\alpha)$, and so G_α is not trivial. G_α clearly has trivial radical, however, and therefore it is semisimple. Let $x \in T_\alpha$. Then, for all $y \in Z_\alpha$, we have $xy = yx$, by definition of Z_α. This in turn gives us $T_\alpha \subset Z(Z_\alpha)$, and since T_α is connected, we have $T_\alpha \subset Z(Z_\alpha)^\circ$. Certainly $Z(Z_\alpha)^\circ \subset R(Z_\alpha)$, by definition of the radical, and so $T_\alpha \subset R(Z_\alpha)$. Therefore, the image of T_α in G_α is trivial, and so

$$\mathrm{rank}\, G_\alpha \leq \dim T - \dim T_\alpha = 1$$

But $\mathrm{rank}\, G_\alpha > 0$, since $G_\alpha \neq 0$. Therefore, $\mathrm{rank}\, G_\alpha = 1$, as required. $\square$

Proposition 3.3.1.5. *Z_α has Weyl group of order 2.*

The proof of Proposition 3.3.1.5 requires the following lemma:

Lemma 3.3.1.6. *Suppose G an algebraic group such that $G/R(G)$ is semisimple and of rank 1. Then the Weyl group $W = N_G(T)/C_G(T)$ is of order 2, and therefore* $\dim G/B = 1$.

Proof. We use Proposition 3.2.2.9, applied to the epimorphism

$$\varphi : G \longrightarrow G' = G/R(G)$$

Then certainly $\ker \varphi = R(G)$ lies in every Borel subgroup of G, and therefore the induced maps $\mathscr{B}^T \to \mathscr{B}'^{T'}$ and $W \to W'$ are bijective. But $T' = \varphi(T) = T/T \cap R(G)$ is of dimension 1, since $G/R(G)$ is assumed to be semisimple and of rank 1, and therefore $T = \mathbb{G}_m$. Now, clearly $N_{G'}(T') \subset \mathrm{Aut}(\mathbb{G}_m)$, and so $\dim W' \leq \dim \mathrm{Aut}(\mathbb{G}_m)$. It is not difficult to show that the only automorphisms of $\mathbb{G}_m$ are $x \mapsto x$ and $x \mapsto x^{-1}$, whence $\mathrm{Aut}(\mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$, and therefore $|W| = |W'| \leq 2$. But, since G is assumed to be nonsolvable, Proposition 3.2.4.7(3) tells us that $W \neq \{e\}$, and therefore $|W| = 2$. The final statement of the lemma now follows from Proposition 3.2.4.7(4) □

Proof of Proposition 3.3.1.5. Proposition 3.3.1.4 says that Z_α satisfies the hypothesis of Lemma 3.3.1.6, and so Z_α has a Weyl group of order 2. □

Lemma 3.3.1.7. *W_α can be identified canonically with a subgroup of W. In particular, the order of W is even.*

Proof. Since $\alpha \in \mathrm{X}(T)$, we certainly have $T_\alpha = (\ker \alpha)^\circ \subset T$ which in turn implies $C_G(T) \subset C_G(T_\alpha) = Z_\alpha$. This then gives us

$$C_{Z_\alpha}(T) = Z_\alpha \cap C_G(T) = C_G(T)$$

and so, since $N_{Z_\alpha}(T) \subset N_G(T)$, we have

$$W_\alpha = N_{Z_\alpha}(T)/C_G(T) \subset N_G(T)/C_G(T) = W$$

The last statement of the lemma follows from Lagrange's Theorem, since W is finite. □

Let W_α be the Weyl group of Z_α. Since W_α has order 2, it follows that T, as a maximal torus of Z_α, lies in precisely two Borel subgroups of Z_α, call them B_α and B'_α.

Example 30. Let $G = SL(n, K)$ again, as above.

For $\alpha \in \Psi$ of the form given above in (3.15), suppose $i < j$. We aim to describe W_α. We begin by describing $N_{Z_\alpha}(T)$. Recall that $N_G(T)$ consists of monomial matrices with determinant 1. It is not difficult to show that an element $n \in N_{Z_\alpha}(T)$ has zero entries everywhere except

1. the k^{th} diagonal entry, where $k \neq i, j$, and
2. the $(i, j)^{\text{th}}$ entry, and

3. the $(j,i)^{\text{th}}$ entry

all of which are nonzero. It follows that W_α is isomorphic to the subgroup $S \subset \mathfrak{S}_n$ which is generated by the transposition (i,j).

We also want to describe the Borel subgroups of Z_α. An element $x \in SL(n,K)$ lies in B_α say, if $x_{rs} = 0$ for all entries except when

1. $r = s$, or

2. $r = i$ and $s = j$.

On the other hand, B'_α consist of elements $x \in SL(n,K)$ with $x_{rs} = 0$ for all entries except when This forces $x_{rs} = 0$ unless

1. $r = s$, or

2. $r = j$ and $s = i$.

3.3.2 The Group $\mathrm{PGL}(2,K)$

The aim of this section is to examine the group $\mathrm{Aut}(\mathbb{P}^1)$. It recalls the group we saw in Example 28, namely $\mathrm{PGL}(2,K) = GL(2,K)/S$. We begin, however, by examining the action of $GL(2,K)$ on $K^2 - \{0,0\}$.

Lemma 3.3.2.1. *Let $v_1, w_1, v_2, w_2 \in K^2 - \{0,0\}$ with $v_1 \neq \lambda_1 w_1$ for all $\lambda_1 \in K^*$ and $v_2 \neq \lambda_2 w_2$ for all $\lambda_2 \in K^*$. Then there exists an element $x \in GL(2,K)$ such that $xv_1 = v_2$ and $xw_1 = w_2$.*

Proof. Since $v_i \neq \lambda_i w_i$ for $i = 1,2$, it follows that (v_1, w_1) and (v_2, w_2) are both bases of K^2. We simply need to take $x \in GL(2,K)$ to be the relevant change of basis matrix. □

Lemma 3.3.2.1 says that the action of $GL(2,K)$ on $K^2 - \{0,0\}$ induces a *doubly transitive* action of $GL(2,K)$ on lines through the origin in K^2, that is, an action on $\mathbb{P}^1$. Of course, the action of scalar matrices S in $GL(2,K)$ leaves lines fixed. It follows that we have an action

$$\begin{aligned} GL(2,K)/S \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) &\longmapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \end{aligned}$$

Since this action is polynomial, it follows that each element of $\mathrm{PGL}(2,K) = GL(2,K)/S$ is a morphism of the variety $\mathbb{P}^1$, and so $\mathrm{PGL}(2,K) \subset \mathrm{Aut}(\mathbb{P}^1)$. In fact, we can say more:

Theorem 3.3.2.2. $\mathrm{PGL}(2,K) = \mathrm{Aut}(\mathbb{P}^1)$.

Proof. Write $G = \mathrm{PGL}(2, K)$. As in §3.2.4, we will construe elements of $\mathbb{P}^1$ as elements of $\mathbf{A}^1 \cup \{\infty\}$, by taking the correspondence $\begin{pmatrix} x \\ 1 \end{pmatrix} \leftrightarrow x$. In particular, we are concerned here with the elements $0, 1, \infty$ of $\mathbf{A}^1 \cup \{\infty\}$, which correspond, respectively, to the elements $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ of $\mathbb{P}^1$.

We define a map

$$\begin{aligned} \phi : G &\longrightarrow \mathbb{P}^3 \\ g &\longmapsto \big(g(0), g(1), g(\infty)\big) \end{aligned}$$

That is,

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} a+b \\ c+d \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}\right) \tag{3.16}$$

It is immediate that ϕ is G-equivariant. Suppose now that $g \in G$ is such that $\phi(g) = (0, 1, \infty)$. A quick examination of (3.16) shows that this implies $g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Suppose now that $h_1, h_2 \in G$ such that $\phi(h_1) = \phi(h_2)$. Then, by G-equivariance,

$$(0, 1, \infty) = \phi(e) = h_1^{-1}\phi(h_1) = h^{-1}\phi(h_2) = \phi(h_1^{-1}h_2)$$

By the above argument, therefore, it follows that $h_1^{-1}h_2 = e$ which means $h_1 = h_2$, and so ϕ is injective.

Note that, for all $g \in G$, the points $g(0), g(1)$ and $g(\infty)$ are distinct, since g is invertible. Suppose that $(x, y, z) \in \mathbb{P}^3$ with pairwise distinct coordinates. Since x, z are linearly independent when considered as elements of K^2, Lemma 3.3.2.1 says that there is an element $g \in G$ such that $gx = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ and $gz = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \infty$. Denote $gy = \begin{pmatrix} u \\ v \end{pmatrix}$. Certainly $gy \neq 0$ and $gy \neq \infty$, since x, y, z are pairwise distinct, and so $u, v \neq 0$.

Write $h = \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix}$, and observe that

$$h(g(x, y, x)) = \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix}\left(0, \begin{pmatrix} u \\ v \end{pmatrix}, \infty\right) = \left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} u^{-1} \\ 0 \end{pmatrix}\right) = (0, 1, \infty)$$

It follows, therefore, that $\phi(g^{-1}h^{-1}) = (x, y, z)$. We have shown that any element in $\mathbb{P}^3$ with pairwise distinct coordinates lies in the image of ϕ.

Let $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$. Since σ is invertible, it follows that the triple $(\sigma(0), \sigma(1), \sigma(\infty))$ is pairwise distinct, and so the above argument shows that there is an element $g \in G$ such that $\phi(g) = (\sigma(0), \sigma(1), \sigma(\infty))$, which means $g(0) = \sigma(0), g(1) = \sigma(1)$ and $\sigma(\infty)$. Consider the morphism $\tau = \sigma^{-1}g$. We know that $G \subset \mathrm{Aut}(\mathbb{P}^1)$, and so τ is an automorphism, and one which fixes the points $(0, 1, \infty)$. In particular, τ restricts to an automorphism of $\mathbf{A}^1$. But since an element of $\mathrm{Aut}(\mathbb{P}^1)$

is necessarily an element of $K[T]$ which is invertible, and whose inverse also lies in $K[T]$, the only possible candidates are morphisms of the form $T \mapsto a + bT$, where $a, b \in K$, and $b \neq 0$. So $\tau(T) = a + bT$, with $0 = \tau(0) = a$, and so $a = 0$, and similarly, $1 = \tau(1) = b(1) = b$, and so $b = 1$. We have thus shown that $\sigma^{-1}g = 1$, and so $\sigma = g \in G$. It follows that $G = \mathrm{Aut}(\mathbb{P}^1)$. □

3.3.3 Semisimple Rank 1 Results

For this section, we will assume that G is connected. As usual, let T be a maximal torus of G, and let $B \subset \mathscr{B}^T$ be a Borel subgroup containing T.

Definition 3.3.3.1. The *rank* of G is the dimension of a maximal torus of G. The *semisimple rank* of G, denoted by $\mathrm{rank}_{ss}(G)$, is the rank of $G/R(G)$. The *reduced rank* of G, denoted by $\mathrm{rank}_{red}(G)$, is the rank of $G/R_u(G)$.

Example 31. $\mathrm{rank}(\mathrm{SL}(n, K)) = n - 1$.

Example 32. $\mathrm{rank}(\mathrm{PGL}(2, K)) = 1$, as seen in Example 28.

Example 33. $\mathrm{rank}_{ss}(GL(2, K)) = 1$. By Corollary 3.1.4.7, a maximal torus of $G/R(G)$ is $D(2, K)/R(G)$. But we saw in Example 20, $R(G) = \mathbb{G}_m$, and so this maximal torus is of dimension 1.

Example 34. $\mathrm{rank}_{red}(GL(n, K)) = n$. By Example 21, $GL(n, K)/R_u(GL(n, K)) = GL(n, K)$, so $\mathrm{rank}_{red}(GL(n, K)) = \mathrm{rank}(GL(n, K)) = n$.

Theorem 3.3.3.2. *Let T be a maximal torus of G, and W its Weyl group. Then the following statements are equivalent:*

1. $\mathrm{rank}_{ss}(G) = 1$
2. $|W| = 2$
3. $\mathrm{Card}(\mathscr{B}^T) = 2$
4. $\dim G/B = 1$
5. $G/B \cong \mathbb{P}^1$
6. *There exists a surjective morphism of varieties*

$$\varphi : G \longrightarrow \mathrm{PGL}(2, K)$$

with $(\ker \varphi)^\circ = R(G)$.

Proof. (1) $\Longrightarrow$ (2): This is by Lemma 3.3.1.6.

(2) $\Longrightarrow$ (3): This is by Proposition 3.2.2.6.

(3) $\Longrightarrow$ (4): This is by Proposition 3.2.4.7(4).

(4) $\Longrightarrow$ (5): Let $\dim G/B = 1$, with G/B embedded in $GL(V)$ in the usual way. Since $G \neq B$, we can find a vector $v \in V$ not fixed by the T-action on

V, and so, just as we saw in Proposition 3.2.4.4, we can take $\lambda \in \mathrm{Y}(T)$, and construct extend it to a non-constant morphism

$$\phi : \mathbb{P}^1 \longrightarrow G/B \subset \mathbb{P}(V).$$

Since $\mathbb{P}^1$ is complete, by Proposition 3.1.2.4 (3) we know that the image of ϕ is closed and complete, and it is certainly irreducible. But G/B is irreducible, too, and of dimension one, and since ϕ is non-constant, $\dim \operatorname{im}(\phi) > 0$, which implies that $\operatorname{im}\phi = G/B$. We cite Theorem 6.3 of [6] at this point, which says that if X is a (smooth) variety of dimension 1 which admits of a surjective morphism $f : \mathbb{P}^1 \to X$, then X is isomorphic to $\mathbb{P}^1$. In our case, G/B is an appropriate candidate (its smoothness coming from the fact that it is a homogeneous G-space), and so we conclude that $\mathbb{P}^1 \cong G/B$, as required.

(5) $\Longrightarrow$ (6): It is clear that each element of G acts as an automorphism of $G/B \cong \mathbb{P}^1$. We saw in Theorem 3.3.2.2 that $\operatorname{Aut}(\mathbb{P}^1) = \operatorname{PGL}(2, K)$, and so we have a map $\varphi : G \to \operatorname{PGL}(2, K)$. If $g \in \ker \varphi$, then $g(xB) = xB$, and so $g \in xBx^{-1}$, for all $x \in G$. It follows that g lies in every Borel subgroup, and so $\ker \varphi$ lies in the intersection of all Borel subgroups. On the other hand, if g lies in the intersection of all Borel subgroups, then $g \in xBx^{-1}$ for all $x \in G$, which means $g(xB) = xB$ for all B, which in turn means that g is a trivial automorphism of $\mathbb{P}^1$. It follows that the intersection of all Borel subgroups is contained in $\ker \varphi$.

By Proposition 3.1.7.2, we have $(\ker \varphi)^\circ = R(G)$. By definition of the radical, $G' = G/R(G)$ is nonsolvable, and therefore, by Corollary 3.1.5.2, $\dim G' > 2$. We saw in Example 28 that $\dim \operatorname{PGL}(2, K) = 3$. Now

$$G/R(G) = \operatorname{im}\varphi \subset \operatorname{PGL}(2, K)$$

and taking dimensions, this becomes

$$3 \leq \dim G/R(G) = \dim(\operatorname{im}\varphi) \leq \dim \operatorname{PGL}(2, K) = 3$$

whence $\dim(\operatorname{im}\varphi) = 3$, and since both groups are connected, $\operatorname{im}\varphi = \operatorname{PGL}(2, K)$, as required.

(6) $\Longrightarrow$ (1): By assumption, $\dim G = \dim R(G) + \dim(\operatorname{PGL}(2, K))$. But if we consider the surjective quotient morphism

$$\alpha : G \longrightarrow G/R(G)$$

then $\dim G = \dim R(G) + \dim(G/R(G))$. It follows that $\dim(G/R(G)) = \dim(\operatorname{PGL}(2, K))$, and since both groups are connected, they are isomorphic. We conclude that $\operatorname{rank}_{ss}(G) = \operatorname{rank}(\operatorname{PGL}(2, K))$ which we saw in Example 28 is equal to 1, as required. □

Corollary 3.3.3.3. *If G is semisimple and* $\operatorname{rank}(G) = 1$, *then* $\dim G = 3$.

Proof. Since G is semisimple, we have $G = G/R(G)$, whence $\operatorname{rank}(G) = \operatorname{rank}_{ss}(G) = 1$, and so we have a surjective morphism of algebraic varieties

$$\varphi : G \longrightarrow \operatorname{PGL}(2, K)$$

Now, $\dim G = \dim \mathrm{im}\varphi + \dim \ker \varphi$, and $\ker \varphi^\circ = R(G) = \{e\}$, whence $\dim \ker \varphi = \dim \ker \varphi^\circ = 0$, and so $\dim G = \dim \mathrm{PGL}(2, K) = 3$. $\square$

Example 35. We saw in Proposition 3.3.1.4 that $G_\alpha = Z_\alpha / R(Z_\alpha)$ is a semisimple group of rank 1. It follows that $3 = \dim G_\alpha = \dim Z_\alpha - \dim R(Z_\alpha)$. Now T is obviously not normal in Z_α, since otherwise it would be the unique maximal torus, and thus by Proposition 3.1.5.1(2) it would be nilpotent and so solvable, which contradicts the fact that T_α is a singular torus. On the other hand, T_α is normal in Z_α, since $Z_\alpha = C_G(T_\alpha)$. It follows, therefore, that T_α is a maximal torus of $R(Z_\alpha)$, and since this latter group is solvable, by Theorem 2.3.4.4 we have $\dim R(Z_\alpha) = \dim T_\alpha + \dim R_u(Z_\alpha)$. Therefore

$$\dim Z_\alpha = \dim R_u(Z_\alpha) + \dim T_\alpha + 3 = \dim R_u(Z_\alpha) + \dim T + 2 \qquad (3.17)$$

Corollary 3.3.3.4. *Let G be a reductive group, with* $\mathrm{rank}_{ss}(G) = 1$. *Let T be a maximal torus of G, and set $Z = Z(G)^\circ$. Then*

1. *(G, G) is semisimple and of dimension 3.*
2. *$G = (G, G) \cdot Z$ and $(G, G) \cap Z$ is finite.*
3. *$C_G(T) = T$ and, in particular, $Z(G) \subset T$.*

Proof. Since G is reduced, we know from Lemma 3.1.1.9 $R(G) = Z$ and is a torus, and, moreover, that $Z \cap (G, G)$ is finite. We now observe that $\mathrm{PGL}(2, K)$ is equal to its own derived group. This is because otherwise, we have

$$3 = \dim \mathrm{PGL}(2, K) > \dim \big(\mathrm{PGL}(2, K), \mathrm{PGL}(2, K)\big)$$

Proposition 2.2.1.3 tells us that $\big(\mathrm{PGL}(2, K), \mathrm{PGL}(2, K)\big)$ is solvable, and therefore that $\mathrm{PGL}(2, K)$ itself is. But $\mathrm{PGL}(2, K)$ is semisimple by an immediate application of Theorem 3.3.3.2(6), and therefore cannot be solvable. Therefore, it must be equal to its derived group.

We apply Theorem 3.3.3.2(6) to give us an epimorphism $\varphi : G \to \mathrm{PGL}(2, K)$ with $(\ker \varphi)^\circ = R(G) = Z$. Recall that $\varphi(G, G) = (\varphi G, \varphi G)$, and so, since φ is an epimorphism, $\varphi(G, G) = \mathrm{PGL}(2, K)$, and so

$$(G, G)/(G, G) \cap Z = \mathrm{PGL}(2, K) \qquad (3.18)$$

But $(G, G) \cap Z$ is finite, and so certainly $\dim(G, G) = 3$, and (G, G) is semisimple, thus proving (1). Now equation (3.18) tells us that

$$G/Z = (G, G)/(G, G) \cap Z$$

and so, for any $x \in G$, we have

$$xZ = y((G, G) \cap Z)$$

for some $y \in (G, G)$. That is, for some $t \in Z, s \in (G, G) \cap Z$, we have $xt = ys$, or $x = y(st^{-1})$. Since $st^{-1} \in Z$, we have shown that $G = (G, G) \cdot Z$, and thus we have proven (2).

Now, any maximal torus T' in $\mathrm{PGL}(2,K)$ is equal to its own centraliser; indeed, supposing T' consists of diagonal matrices, if $t \in T'$ and $x \in C_{\mathrm{PGL}(2,K)}(T')$, then

$$(txt^{-1})_{ij} = t_i/t_j x_{ij}$$

So, if $i \neq j$, and $t_i \neq t_j$, then $x_{ij} = 0$, and so it follows that x_{ij} is diagonal, and so lies in T'.

Now, it is easy to check that $\varphi(C_G(T)) \subset C_{\varphi(G)}(\varphi(T))$. But then, by the above argument, and since $T \subset C_G(T)$, we have

$$\varphi(T) \subset \varphi(C_G(T)) \subset C_{\varphi(G)}(\varphi(T)) = \varphi(T)$$

Therefore, $\varphi(T) = \varphi(C_G(T))$. Now, given an element $x \in C_G(T)$, we apply φ to see that $\varphi(x) \in \varphi(C_G(T)) = \varphi(T)$, and therefore $\varphi(x) = \varphi(t)$ for some $t \in T$. In turn, we have $t^{-1}x \in \ker\varphi$, and therefore $x \in T \cdot \ker\varphi$. So we have

$$T \subset C_G(T) \subset T \cdot \ker\varphi \tag{3.19}$$

Now, we know that $\dim(\ker\varphi) + \dim \mathrm{PGL}(2,K) = \dim G$. But $\dim G \geq \dim(G,G) = 3$, by (1), and $\dim \mathrm{PGL}(2,K) = 3$, which means $\dim(\ker\varphi) = 0$, and so $\ker\varphi$ is finite. This then implies that T has finite index in $T \cdot \ker\varphi$, and therefore by Proposition 1.3.1.1(5), we have $(T \cdot \ker\varphi)^\circ \subset T$. But since T is connected, equation (3.19) tells us $T \subset (T \cdot \ker\varphi)^\circ$, and so $T = (T \cdot \ker\varphi)^\circ$. But Theorem tconndtor2 tells us that $C_G(T)$ is connected, and so the same argument as above shows that $C_G(T) = (T \cdot \ker\varphi)^\circ$, and so, it follows that $T = C_G(T)$, whence (3). □

3.4 Structure of Reductive Groups

In this section we finally advance the structure theory that we have been waiting so long for. In fact, the whole theory is in a sense a series of corollaries, extending from the first result of this section, namely Theorem 3.4.1.1. It is fascinating to watch the seemingly never ending cascade of results which fall out from this result.

3.4.1 The Unipotent Radical

Recall, from §3.3.1 the notation:

$$I(T) = \Big(\bigcap_{B \in \mathscr{B}^T} B\Big)^\circ$$

Theorem 3.4.1.1. $R_u(G) = I(T)_u$

The proof of Theorem 3.4.1.1 requires the following lemmas:

Lemma 3.4.1.2. *Let T be a maximal torus of G and let $B \in \mathscr{B}^T$. Then B is generated by the subgroups $C_B(S)$, where $S \subset T$ is a subtorus of codimension 1 in T.*

Proof. Let A be the subgroup generated by the subgroups $C_B(S)$ with S ranging over the subtori of T with $\dim S = \dim T - 1$. Then certainly $A \subset B$, and so $\mathscr{L}(A) = \mathfrak{a} \subset \mathfrak{b} = \mathscr{L}(B)$. We now consider the decomposition of $\mathfrak{b}$ into weight spaces $\mathfrak{g}_\alpha$ with respect to T. In the first place, suppose x is of weight 0, that is, $\mathsf{x} \in \mathfrak{c}_\mathfrak{b}(T) = \mathscr{L}(C_B(T))$, where the equality is by Proposition 2.3.1.9 . Since $C_B(T) \subset C_B(S)$, we have $\mathscr{L}(C_B(T)) \subset \mathscr{L}(C_B(S))$. But $C_B(S) \subset A$, and so $\mathscr{L}(C_B(S)) \subset \mathfrak{a}$, which in turn means that $\mathsf{x} \in \mathfrak{a}$. Now suppose that $\alpha \in \Phi(B,T)$ is a nonzero weight, with $\mathsf{x} \in \mathfrak{g}_\alpha$. Consider $T_\alpha = (\ker\alpha)^\circ$. Certainly S is a torus of codimension 1 in T, and so $C_B(T_\alpha) \subset A$. If $t \in T_\alpha \subset T$, then

$$t\mathsf{x}t^{-1} = \alpha(t)\mathsf{x} = \mathsf{x}$$

and so

$$\mathsf{x} \in \mathfrak{c}_\mathfrak{b}(T_\alpha) = \mathscr{L}(C_B(T_\alpha)) \subset \mathfrak{a}$$

where the middle equality is again by Proposition 2.3.1.9. We have shown that $\mathfrak{b} \subset \mathfrak{a}$, and so, combining this with our earlier inclusion, we have $\mathfrak{a} = \mathfrak{b}$. This implies that $\dim A = \dim B$.

Now, A is connected. This is because each centraliser $C_B(S)$ is connected by Theorem 3.1.6.5, and so $C_B(S) \subset A^\circ$. But since A is minimal with respect to these inclusions, it follows that $A^\circ = A$. Certainly B is connected, and so $A \subset B$, combined with the equality $\mathfrak{a} = \mathfrak{b}$, gives $A = B$, as required. □

Lemma 3.4.1.3. *Let B be a Borel subgroup of G, containing a maximal torus T. If $S \subset T$ is a subtorus of codimension 1 in T, then $C_B(S) \subset N_G(I(T)_u)$*

Proof. For convenience, we will write $U = I(T)_u$ and $\mathfrak{u} = \mathscr{L}(U)$. Suppose firstly that S is regular. Then we saw in Proposition 3.2.1.2 that, for any Borel subgroup of G which contains T, $C_B(S) = C_G(S)$, and that $C_G(S) \subset B$. Since $C_G(S)$ is connected, by Theorem 3.1.6.5, $C_G(S) \subset I(T)$. Certainly, if $s \in C_G(S)$, then $sxs^{-1} \in I(T)$. Since sxs^{-1} is unipotent if x is, $sxs^{-1} \in U$ if $x \in U$, as required.

Suppose now that S is singular. By Proposition 3.3.1.2, $S = (\ker\alpha)^\circ = T_\alpha$ for some $\alpha \in \Psi$. Now, by Corollary 3.1.6.8, $C_B(T_\alpha) = Z_\alpha \cap B$ is one of the two Borel subgroups of Z_α. Note, moreover, that $R_u(Z_\alpha) \subset U$, and since $\dim Z_\alpha/(R_u(Z_\alpha)) = \dim T + 2$ by equation (3.17), it is clear that $\pm\alpha$ are the only roots of $\dim Z_\alpha/(R_u(Z_\alpha))$, and therefore $\pm\alpha$ are the only roots of Z_α which lie outside U, and they occur with multiplicity one.

Define a subgroup H of G as follows:

$$H = \bigcap_{\substack{B\in\mathscr{B}^T \\ \alpha \text{ is a root of } Z_\alpha\cap B}} {B_u}^\circ$$

Note that $U \subset H \subset (B_\alpha)_u$. We wish to show that U is normal in H. Now certainly H is T-stable, so we can decompose

$$\mathfrak{h} = \mathfrak{u} \oplus \bigoplus_{\beta\in\Omega} \mathfrak{g}'_\beta$$

where $\Omega = \{\beta \in \Psi \mid \mathfrak{u} \cap \mathfrak{g}_\beta \neq 0\}$ that is, Ω consists of the roots of H outside U. Note in the first place that $\alpha \in \Omega$. Moreover, we showed above that if $\beta \in \Omega$, then the only multiples of β which are roots of Z_β which lie outside of U are $\pm\beta$. But $H \subset B$, so if $\beta \in \Omega$, then $-\beta \notin \Omega$.

It can be shown using a purely geometric argument such as that of Lemma 26.1 of [6] that, given distinct, non-proportional roots $\alpha, \beta \in \Psi$, we can find a Borel subgroup B' such that $B' \cap Z_\alpha$ has α as a root, but does not have β as a root. (In fact Lemma 26.1 of [6] expresses the result in terms of the pairing $\langle \cdot, \cdot \rangle$ on $\mathrm{X}(T)$ and $\mathrm{Y}(T)$. The geometric connection is arises from the relation between this dual pairing and the Weyl chamber of a Borel subgroup, a notion which we discuss further in §3.5.2.) We just argued above that if $\beta \in \Omega$ is distinct from α, then α and β are non-proportional, so nominate such a Borel subgroup B'. By construction, $H \subset B'$.

Take $\mathsf{x} \in \mathfrak{g}'_\beta$. Then $\mathsf{x} \in \mathfrak{c}_\mathfrak{h}(T_\beta) = \mathscr{L}(C_H(T_\beta))$, where this latter equality is by Proposition 2.3.1.9. But $C_H(T_\beta) = H \cap Z_\beta = H \cap B' \cap Z_\beta = \{e\}$, where this last equality is by construction of B'. Thus $0 = \dim C_H(T_\beta) = \dim \mathfrak{c}_\mathfrak{h}(T_\beta)$, and so $\mathsf{x} = 0$. But this then means $\beta \notin \Omega$. It follows, then, that α is the only root of H which lies outside U. We saw above that $\dim \mathfrak{g}'_\alpha = 1$, and so $\dim H = \dim U + 1$. Since H is unipotent, and therefore nilpotent, we can apply Proposition 2.2.3.6(2), which says that U is normal in H. Since H is contained in $(B_\alpha)_u$, the result follows. □

Proof of Theorem 3.4.1.1. We need to show that $R_u(G) = I(T)_u$. By Lemma 3.1.7.2, $R_u(G)$ is the intersection of the unipotent subgroups of all Borel subgroups, so it is immediate that $R_u(G) \subset I(T)_u$. On the other hand, $I(T)_u$ is unipotent and connected, and so we just need to show that it is normal in G for the reverse inclusion to hold. Now Lemma 3.4.1.3 tells us that $I(T)_u$ is normalised by all $C_B(S)$ where $S \subset T$ is a subtorus of codimension one, and $B \in \mathscr{B}^T$. In turn, Lemma 3.4.1.2 tells us that each $B \in \mathscr{B}^T$ is generated by all such centralisers, and so $I(T)_u$ is normalised by B. Finally, Proposition 3.2.4.7(5) tells us that G is generated by the elements of $\mathscr{B}^T$, and so we can conclude that $I(T)_u$ is normalised by all of G, as required. □

Corollary 3.4.1.4. $I(T) = T \cdot R_u(G)$

Proof. Since $I(T)$ is a subgroup of B, it is solvable. Moreover, $T \subset I(T)$, and so T is a maximal torus of $I(T)$. Applying Theorem 2.3.4.4 combined with Theorem 3.4.1.1 gives us the result. □

3.4.2 First Theorem for Reductive Groups

Corollary 3.4.2.1. *Let G be reductive, and S any subtorus of T. Then*

1. *$C_G(S)$ is reductive.*

2. *If S is regular, then $C_G(S) = T$. In particular, the Cartan subgroups of G are just the maximal tori, that is, for any maximal torus T' of G, we have $C_G(T')^\circ = T'$.*

3. $Z(G) \subset T$.

Proof. For (1), apply Theorem 3.4.1.1 to the group $C_G(S)$. T is again a maximal torus of $C_G(S)$, which tells us

$$R_u(C_G(S)) = \Big(\bigcap_{\substack{T \subset B \subset C_G(S) \\ B \text{ is Borel in } C_G(S)}} B \Big)_u \tag{3.20}$$

By Corollary 3.1.6.8, we know that the Borel groups B of $C_G(S)$ in the above equation are of the form $C_G(S) \cap B'$, where B' is Borel in G. This tells us that the left hand side of equation (3.20) lies in $I(T)_u$. But applying Theorem 3.4.1.1 again, this time to G, we see that

$$R_u(C_G(S)) \subset I(T)_u = R_u(G) = \{e\}$$

and so $R_u(C_G(S)) = \{e\}$, that is, $C_G(S)$ is reductive.

Now (2) quickly follows, since, by (1) we know that $C_G(S)$ is reductive, and by Proposition 3.2.1.2, since S is assumed regular it follows that $C_G(S)$ is solvable. Since S is a torus, Theorem 3.1.6.5tells us that $C_G(S)$ is connected. Furthermore, Lemma 3.1.1.9 says that $R(C_G(S))$ is a torus, but since $C_G(S)$ is solvable and connected, we have $R(C_G(S)) = C_G(S)$, and therefore $C_G(S)$ is a torus. But $T \subset C_G(S)$, with T maximal, therefore $T = C_G(S)$, as required. Finally, (3) follows immediately, when we observe the simple fact that $Z(G) \subset C_G(T)$, the latter being equal to T by (2). ☐

3.4.3 Second Theorem for Reductive Groups

Lemma 3.4.3.1. *Let G be a group with roots α, β. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$*

Proof. Let $\mathsf{x} \in \mathfrak{g}_\alpha, \mathsf{y} \in \mathfrak{g}_\beta$, and let $t \in T$. Then

$$\begin{aligned} \mathrm{Ad}t([\mathsf{x}, \mathsf{y}]) &= t(\mathsf{xy} - \mathsf{yx})t^{-1} \\ &= t\mathsf{x}t^{-1}t\mathsf{y}t^{-1} - t\mathsf{y}t^{-1}t\mathsf{x}t^{-1} \\ &= \alpha(t)\mathsf{x}\beta(t)\mathsf{y} - \beta(t)\mathsf{y}\alpha(t)\mathsf{x} \\ &= \big(\alpha(t) \cdot \beta(t)\big)[\mathsf{x}, \mathsf{y}] \\ &= (\alpha + \beta)(t)[\mathsf{x}, \mathsf{y}] \end{aligned}$$

which implies $[\mathsf{x}, \mathsf{y}] \subset \mathfrak{g}_{\alpha+\beta}$, as required. ☐

Theorem 3.4.3.2. *Let G be reductive, and $\Phi = \Phi(G, T)$. Then*

1. $\Phi = \Psi$.

2. *The singular tori of codimension 1 in G are the tori $T_\alpha = (\ker \alpha)^\circ$ for $\alpha \in \Phi$.*

3. *Let $Z_\alpha = C_G(T_\alpha)$, for $\alpha \in \Phi$. Then Z_α is a reductive group of semisimple rank 1, with*

$$\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

Moreover, the groups Z_α generate G.

4. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

5. $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, *and* $\dim \mathfrak{g}_\alpha = 1$.

6. $Z(G)^\circ = \left(\bigcap_{\alpha\in\Phi} T_\alpha\right)^\circ$.

7. *If $R \subset \mathrm{X}(G)$ is the group generated by Φ, then* $\operatorname{rank}(R) = \operatorname{rank}_{ss}(G)$.

Proof. In equation (3.13), we used the decomposition

$$\mathfrak{g} = \mathscr{L}(I(T)) \oplus \coprod_{\alpha\in\Psi} \mathfrak{g}_\alpha$$

Now Corollary 3.4.1.4 tells us that $I(T) = T \cdot R_u(G) = T$, since G is reductive. But Corollary 3.4.2.1(2) tells us $T = C_G(T)$, and so $\Phi = \Psi$, by Lemma 3.3.1.1.

The proof of (2) is now immediate, since if S is a singular subtorus of T of codimension 1, then Proposition 3.3.1.2 tells us that $S \subset T_\alpha$ for some $\alpha \in \Psi = \Phi$.

We now move on to proving (3). By Corollary 3.4.2.1(1), $Z_\alpha = C_G(T_\alpha)$ is reductive. But then Lemma 3.1.1.9 says that $R(Z_\alpha)$ is a torus. Certainly $R(Z_\alpha) \neq T$, since this would imply T is normal, and therefore that it is the unique maximal torus of Z_α. But we saw in Proposition 3.1.5.1(2) that this implies Z_α is nilpotent and so solvable, which is contrary to the fact that T_α is singular. However, T_α is certainly a normal torus of Z_α, and being of codimension 1 in T it is the maximal such torus. Therefore $R(Z_\alpha) = T_\alpha$. Now, we saw in Theorem 3.3.3.2(6) that $\dim Z_\alpha = \dim R(Z_\alpha) + \dim \mathrm{PGL}(2, K)$. Therefore $\dim Z_\alpha = \dim T - 1 + 3 = \dim T + 2$. Moreover, Theorem 3.3.3.2 also tells us that Z_α has precisely two distinct Borel subgroups, which in turn implies that it must have more than one root, which facts together means that Z_α has precisely two roots.

Certainly $T \subset Z_\alpha$, and so $\mathfrak{t} \subset \mathscr{L}(Z_\alpha)$. Since α is a root of G, we can find an element $\mathsf{x} \in \mathfrak{g}_\alpha$. We claim that $\mathsf{x} \in \mathscr{L}(Z_\alpha)$. Indeed, let $t \in T_\alpha$. Then

$$t\mathsf{x}t^{-1} = \alpha(t)\mathsf{x} = \mathsf{x}$$

since $\alpha(t) = 1$. It follows that $\mathsf{x} \in \mathfrak{c}_\mathfrak{g}(T_\alpha)$. But $\mathfrak{c}_\mathfrak{g}(T_\alpha) = \mathscr{L}(C_G(T_\alpha))$ by Proposition 2.3.1.9. But this latter group is simply $\mathscr{L}(Z_\alpha)$, and so $\mathsf{x} \in \mathscr{L}(Z_\alpha)$. This tells us that α is a root of Z_α relative to T.

Assume now that β is the other root. Let $\mathsf{x} \in \mathfrak{g}_\alpha, \mathsf{y} \in \mathfrak{g}_\beta$. By Lemma 3.4.3.1, $[\mathsf{x}, \mathsf{y}] \in \mathfrak{g}_{\alpha+\beta}$. This implies $\mathfrak{g}_{\alpha+\beta} \neq 0$. But Z_α has only two roots, and therefore $\alpha + \beta$ must be equal to α, β or 0. If $\alpha + \beta = \alpha$, then $\beta = 0$, and therefore β is not a root, contrary to our assumption. Similarly, if $\alpha + \beta = \beta$, then $\alpha = 0$ is not a root, which is impossible. Therefore $\alpha + \beta = 0$, which implies $\beta = -\alpha$.

We now use (3.10) to decompose Z_α, noting that $C_{Z_\alpha}(T) = C_G(T) \cap Z_\alpha = T$. The decomposition thus becomes:

$$\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \tag{3.21}$$

as required.

Suppose now that H is the closed group generated by all Z_α. Then $\mathfrak{g}_\alpha \subset \mathscr{L}(H)$ for all $\alpha \in \Phi$, and so equation (3.10) shows that $\mathfrak{g} \subset \mathscr{L}(H)$. It follows that $\dim H = \dim G$, with $H \subset G$ both connected, and therefore $H = G$, proving (3). Note also that the decomposition in (3.21) says $\mathfrak{g}_{-\alpha}$ is not trivial, and so $-\alpha \in \Phi$, proving (4).

For (5), note that we already have the desired decomposition given as equation (3.10), since in this case $C_G(T) = T$. So we just need to show $\dim(\mathfrak{g}_\alpha) = 1$. For this we use (3.21) again, and in particular the fact that this implies $\dim Z_\alpha = \dim T + \dim(\mathfrak{g}_\alpha) + \dim(\mathfrak{g}_{-\alpha})$. But we have already seen that $\dim Z_\alpha = \dim T + 2$, and so $\dim(\mathfrak{g}_\alpha) + \dim(\mathfrak{g}_{-\alpha}) = 2$, forcing $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$, proving (5).

Suppose now that $t \in T_\alpha$ for all $\alpha \in \Phi$. Then $gt = tg$ for any $g \in Z_\alpha$ for all $\alpha \in \Phi$. But these groups generate G, by (3), so $gt = tg$ for all $g \in G$, whence $Z(G)^\circ \supset \left(\bigcap_{\alpha\in\Phi} T_\alpha\right)^\circ$. On the other hand, $Z(G) \subset T$ by Corollary 3.4.2.1(3), and so is a subtorus of T. Moreover, $C_G(Z(G)) = G$ and so

$$\mathfrak{g} = \mathscr{L}(C_G(Z(G))) = \mathfrak{c}_{\mathfrak{g}}(Z(G))$$

where this last equality is by Corollary 2.3.1.10. It follows, then, that for al $\alpha \in \Phi$, if $\mathsf{x} \in \mathfrak{g}_\alpha \subset \mathfrak{g}$ and $s \in Z(G)$, we have

$$\mathsf{x} = s\mathsf{x}s^{-1} = \alpha(s)\mathsf{x}$$

whence $s \in \ker(\alpha)$. Thus (6) follows.

Finally, for (7), recall that, since $R(G) \subset I(T)$, the roots of G coincide with the roots of $G/R(G)$, and so we can assume that G is semisimple here. Recall also that the $\mathrm{X}(T)$ and $\mathrm{Y}(T)$ are dual $\mathbb{Z}$-modules. Let $R' \subset \mathrm{Y}(T)$ be such that $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Phi, \lambda \in R'$. This implies that $\lambda(\mathbb{G}_m) \subset T_\alpha$ for all $\alpha \in \Phi$, and so by (6), $\lambda(\mathbb{G}_m) \subset Z(G)^\circ$, but this latter group is trivial since G is assumed to be semisimple. Therefore R' is trivial, which means $\mathrm{rank}(R) = \mathrm{rank}\mathrm{X}(T)$, as required. □

3.4.4 The Groups U_α

We assume throughout this section that G is reductive.

Definition 3.4.4.1. For each $\alpha \in \Phi$, let U_α be the unipotent part of one of the two Borel subgroups of the group Z_α, such that $\mathscr{L}(U_\alpha) = \mathfrak{g}_\alpha$.

Example 36. Take $G = SL(n, K)$ again. In example (29) we saw that for the root

$$\alpha : \mathrm{diag}(t_1, \ldots, t_{l+1}) \longmapsto t_i/t_j$$

the group Z_α consists of determinant 1 matrices with non-zero diagonal entries, as well as arbitrary entries in the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ positions, with all other entries equal to zero. Example (30) showed that the Borel subgroups of Z_α are the maximal upper and lower triangular subgroups of Z_α, call them B^+ and B^-, respectively.

It is easy to show that $\mathscr{L}(B^+)$ consists of matrices which have an arbitrary entry in the $(i,j)^{\text{th}}$ position, with zeroes everywhere else. Moreover, for $\mathsf{x} \in \mathscr{L}(B^+)$, it is easy to see that, given $t = \mathrm{diag}(t_1, \ldots, t_{l+1}) \in T$,

$$(t\mathsf{x}t^{-1})_{rs} = t_r/t_s(\mathsf{x})_{rs}$$

which, ultimately, shows $\mathrm{Ad}t(\mathsf{x}) = \alpha(t)\mathsf{x}$, and therefore $\mathscr{L}(B^+) \subset \mathfrak{g}_\alpha$. But $\mathfrak{g}_\alpha$ is 1-dimensional, and therefore this inclusion is in fact an equality. It follows, therefore, that $U_\alpha = \mathscr{L}(B^+)$, and therefore consists of matrices which have an arbitrary entry in the $(i,j)^{\text{th}}$ position, with zeroes everywhere else.

Proposition 3.4.4.2. *U_α is the unique connected, T-stable subgroup of G having Lie algebra $\mathfrak{g}_\alpha$.*

Proof. Of course $T \subset B_\alpha$, where B_α is the Borel subgroup of Z_α which has U_α as unipotent part. Therefore, $B_\alpha = TU_\alpha = U_\alpha T$, and so U_α is T-stable.

Suppose V is a connected, T-stable subgroup such that $\mathscr{L}(V) = \mathfrak{g}_\alpha$. Certainly, then, V is 1 dimensional, and therefore isomorphic either to $\mathbb{G}_a$ or $\mathbb{G}_m$. Now, each element $\mathsf{x} \in \mathfrak{g}_\alpha$ is nilpotent, which means $V \cong \mathbb{G}_a$, and therefore V is unipotent.

Define a new group $H = TV$. Then H is connected, since T and V are. Moreover, since T and V are solvable (the latter being unipotent), Lemma 2.2.2.5 tells us that H must be, too. Being solvable and connected, it follows that H must lie in some $B \in \mathscr{B}^T$.

Since V is assumed to be T-stable, it follows that $H = TV = VT$, and therefore, by exponentiating, say, we see that $\mathscr{L}(H) = \mathfrak{h} = \mathfrak{t} + \mathfrak{g}_\alpha$. Suppose $t \in T_\alpha$. Then, for all $\mathsf{x} \in \mathfrak{g}_\alpha$, we have

$$t\mathsf{x}t^{-1} = \alpha(t)\mathsf{x} = \mathsf{x} \tag{3.22}$$

Moreover, suppose $\mathsf{y} \in \mathfrak{t}$. We know

$$\mathfrak{t} = \mathfrak{c}_\mathfrak{g}(T) \subset \mathfrak{c}_\mathfrak{g}(t) = \{\mathsf{x} \in \mathfrak{g} \mid \mathrm{Ad}t(\mathsf{y}) = \mathsf{y}\}$$

Therefore $t\mathsf{y}t^{-1} = \mathrm{Ad}t(\mathsf{y}) = \mathsf{y}$. This, combined with equation (3.22) tells us

$$t \in T_\alpha \Rightarrow \mathrm{Ad}t(\mathsf{x}) = \mathsf{x},\ \forall \mathsf{x} \in \mathfrak{h}$$

which in turn gives

$$\mathscr{L}(H) = \mathfrak{h} \subset \mathfrak{c}_\mathfrak{g}(T_\alpha) = \mathscr{L}(C_G(T_\alpha))$$

and so $H \subset C_G(T_\alpha) = Z_\alpha$. But $V \subset H$, so $V \subset Z_\alpha$.

Moreover, $V \subset H \subset B$, whence $V \subset B \cap Z_\alpha$. But this intersection necessarily describes one of the Borel subgroups of Z_α, which we may describe as B_α and $B_{-\alpha}$.

Suppose $V \subset B_{-\alpha}$, where $\mathscr{L}\big((B_{-\alpha})_u\big) = \mathfrak{g}_{-\alpha}$. Therefore

$$\mathscr{L}(V) = \mathfrak{g}_\alpha \subset \mathfrak{g}_{-\alpha}$$

which is a contradiciton. Therefore, $V \subset B_\alpha$. But U_α is the unipotent part of B, and since V is itself unipotent, we have $V \subset U_\alpha$. But both U_α and V are connected, and of dimension 1. It follows that $V = U_\alpha$, as required. □

3.4.5 Third Theorem for Reductive Groups

Theorem 3.4.5.1. *Let G be reductive, $\alpha \in \Phi$. Then*

1. *There exists a unique, connected T-stable subgroup U_α of G having Lie algebra $\mathfrak{g}_\alpha$, and $U_\alpha \subset Z_\alpha$*
2. *If $n \in N$ represents $\sigma \in W$, then $nU_\alpha n^{-1} = U_{\sigma(\alpha)}$*
3. *There exists an isomorphism $\epsilon_\alpha : \mathbb{G}_a \to U_\alpha$ such that, for all $t \in T$, $x \in \mathbb{G}_a$, we have*
$$t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x)$$
4. *G is generated by the groups U_α, $\alpha \in \Phi$, and T.*

Proof. We proved (1) in Proposition 3.4.4.2. For (2), note

$$\mathfrak{g}_{\sigma(\alpha)} = \mathrm{Ad}n\mathfrak{g}_\alpha = \mathrm{Ad}n\big(\mathscr{L}(U_\alpha)\big) = \mathscr{L}(nU_\alpha n^{-1})$$

Now, $nU_\alpha n^{-1}$ is certainly connected, and, since U_α is T-stable, for all $t \in T$ we have

$$t(nU_\alpha n^{-1})t^{-1} = n(n^{-1}tnU_\alpha n^{-1}t^{-1}n)n^{-1} = nU_\alpha n^{-1}$$

since $n^{-1}tn \in T$, so nUn^{-1} is itself T-stable. Therefore, by (1), $\mathfrak{g}_{\sigma(\alpha)} = \mathscr{L}(nU_\alpha n^{-1})$ implies $nUn^{-1} = \mathfrak{g}_{\sigma(\alpha)}$.

For (3), note first that, since U_α is unipotent and 1-dimensional, there exists some isomorphism

$$\epsilon : \mathbb{G}_a \longrightarrow U_\alpha$$

We can now construct a morphism of algebraic groups

$$\begin{aligned} \lambda : T &\longrightarrow \mathrm{Aut}\mathbb{G}_a \\ t &\longmapsto \lambda(t) \end{aligned}$$

where $\lambda(t)(x) = \epsilon^{-1}\big(t\epsilon(x)t^{-1}\big)$ for $x \in \mathbb{G}_a$. Now, recall $\mathrm{Aut}\mathbb{G}_a \cong \mathbb{G}_m$, and in particular,

$$\begin{aligned} f : \mathrm{Aut}\mathbb{G}_a &\longrightarrow \mathbb{G}_m \\ \beta &\longmapsto \beta(x)/x, \text{ for any } x \in \mathbb{G}_a \end{aligned}$$

is such an isomorphism. We can compose these maps to get a morphism $\gamma = f \circ \lambda$, that is, $\gamma : T \to \mathbb{G}_m$, and so γ is really a character of T. In summary,

$$\gamma(t) = f \circ \lambda(t) = f\big(x \mapsto \epsilon^{-1}(t\epsilon(x)t^{-1})\big) = \epsilon^{-1}(t\epsilon(x)t^{-1})/x$$

and so

$$\epsilon(\gamma(t)x) = t\epsilon(x)t^{-1}$$

for all $x \in \mathbb{G}_a$. Rephrasing the above, we get an equality of maps $\mathbb{G}_a \to U_\alpha$:

$$\mathrm{Int}t \circ \epsilon(x) = \gamma(t) \cdot \epsilon(x) \tag{3.23}$$

We differentiate both sides of equation (3.23). Note firstly that $d\epsilon = 1$, by (1). Therefore,

$$d\big(\mathrm{Int}t \circ \epsilon\big)(\mathsf{x}) = \mathrm{Ad}t(\mathsf{x}) = d(\gamma(t) \cdot \epsilon)(\mathsf{x}) = \gamma(t)(d\epsilon)(\mathsf{x}) = \gamma(t)\mathsf{x}$$

that is, $\mathrm{Ad}t(\mathsf{x}) = \gamma(t)\mathsf{x}$, which forces $\gamma = \alpha$, and so we write $\epsilon = \epsilon_\alpha$, and we are done.

Finally, (4) follows from Theorem 3.4.3.2(3), which states that the groups Z_α generate G, and the fact that each Z_α is generated by the groups T and U_α. □

Corollary 3.4.5.2. *Let G be a reductive group with a maximal torus T, and let $B \in \mathscr{B}^T$. Then*

$$\mathscr{L}(B) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(B)} \mathfrak{g}_\alpha$$

where $\Phi(B) \subset \Phi$ is such that $\Phi = \Phi(B) \cup -\Phi(B)$. In particular, $\dim B = \mathrm{rank}(G) + \frac{1}{2}\mathrm{Card}(\Phi)$.

Proof. By equation (3.10), we can decompose $\mathscr{L}(B)$ as follows:

$$\mathscr{L}(B) = \mathfrak{c}_{\mathscr{L}(B)}(T) \oplus \bigoplus_{\alpha \in \Phi(B)} \mathscr{L}(B)_\alpha$$

But G is reductive, and so $T = C_G(T)$ by Corollary 3.4.2.1(2). Therefore $C_B(T) = C_G(T) \cap B = T$, and so $\mathfrak{c}_{\mathscr{L}(B)}(T) = \mathscr{L}(C_B(T)) = \mathfrak{t}$. Moreover, for each $\alpha \in \Phi(B)$, we have $\mathscr{L}(B)_\alpha \subset \mathfrak{g}_\alpha$, but by Theorem 3.4.3.25, $\dim \mathfrak{g}_\alpha = 1$, and so $\mathscr{L}(B)_\alpha = \mathfrak{g}_\alpha$.

It remains to show that $\Phi = \Phi(B) \cup -\Phi(B)$. By Theorem 3.4.3.2(3), for each $\alpha \in \Phi$, the roots of Z_α are $\pm\alpha$. Since Z_α is of semisimple rank 1, by Theorem 3.3.3.2 it has precisely two Borel subgroups, one of which is $Z_\alpha \cap B$. It follows that $Z_\alpha \cap B$ contains exactly one of either α or $-\alpha$ as a root, and so the same is true of B. That is, for each $\alpha \in \Phi$, we have either $\alpha \in \Phi(B)$ or $-\alpha \in \Phi(B)$, and so we get the required partition of Φ. □

3.5 Root Systems Again

The purpose of this section is to demonstrate that the characters which we have called *roots* are in fact abstract roots, and therefore we can bring in a whole new swathe of results, inherited from the well developed theory of abstract root systems. Indeed, the classification problem for reductive groups becomes a familiar project after this, as we will see in Chapter 5 that the Classical Groups exhaust all the classes of infinite root systems, and therefore we can be confident that whatever reductive groups are left either correspond to the exceptional, finite root systems, or are quotients or products or the like, built up from the Classical Groups.

3.5.1 Abstract Root Systems

Definition 3.5.1.1. Let E be a finite dimensional vector space over $\mathbb{R}$. An *abstract root system* in E is a subset Ψ of E which satisfies the following axioms:

1. Ψ is finite, spans E, and does not contain 0.
2. If $\alpha \in \Psi$, the only scalar multiples $k\alpha \in \Psi$ are $\pm\alpha$.
3. If $\alpha \in \Psi$, there exists a unique reflection τ_α relative to α which leaves Ψ stable.
4. If $\alpha, \beta \in \Psi$, then $\tau_\alpha(\beta) - \beta$ is an integral multiple of α, not necessarily contained in Ψ.

The *rank* of Φ is defined to be $\dim E$. The *abstract Weyl group* $W(\Phi)$ is the (finite) subgroup of $GL(E)$ generated by all the reflections τ_α for $\alpha \in \Phi$.

We hope to show that the root system $\Phi(G, T)$ of a semisimple algebraic group is also an abstract root system, so we return now to the situation of algebraic groups.

Lemma 3.5.1.2. $\ker \mathrm{Ad} \subset Z(Z_\alpha)$.

Proof. We know from Theorem 3.3.3.2 that there exists an epimorphism

$$\varphi : Z_\alpha \longrightarrow \mathrm{PGL}(2, K)$$

such that $\ker \varphi = Z(Z_\alpha)$ Now, $\ker \mathrm{Ad}$ is a closed, normal subgroup of Z_α, and therefore $\varphi(\ker \mathrm{Ad})$ is a closed, normal subgroup of $\mathrm{PGL}(2, K)$.

Let $\mathsf{x} \in \mathfrak{g}_\alpha$. Then, for each $t \in T$, we have

$$\mathrm{Ad}t(\mathsf{x}) = t\mathsf{x}t^{-1} = \alpha(t)\mathsf{x}$$

which means $\mathrm{Ad}t \neq 1$, and therefore $t \notin \ker \mathrm{Ad}$. This in turn shows that $\dim(\ker \mathrm{Ad}) < \dim Z_\alpha$, and so $\dim(\varphi(\ker \mathrm{Ad})) < \dim \mathrm{PGL}(2, K) = 3$. But Corollary 3.1.5.2 us that all groups of dimension less than or equal to 2 are necessarily solvable. So $\varphi(\ker \mathrm{Ad})$ is solvable, as is $\mathrm{PGL}(2, K)/\varphi(\ker \mathrm{Ad})$, if

$\varphi(\ker \mathrm{Ad})$ is nontrivial. Applying Lemma 2.2.2.5(2), we see that $\mathrm{PGL}(2, K)$ is solvable, which is a contradiction. Therefore $\varphi(\ker \mathrm{Ad})$ is trivial, which, in turn, tells us $\ker \mathrm{Ad} \subset \ker \varphi = Z(Z_\alpha)$, as required. □

Proposition 3.5.1.3. *Let $\rho : G \to GL(V)$ be a rational representation with $\ker \rho \subset Z(G)$, and let V_λ be a weight space relative to $\rho(T)$. If $\alpha \in \Phi$, then $\rho(U_\alpha)$ sends V_χ to $\sum V_{\chi + k\alpha}$ where $k \in \mathbb{Z}^+$.*

Proof. Write $G' = \rho(G)$. Firstly note that $Z(G)$ lies in all Borel subgroups of G, since all Borel subgroups are conjugate. It therefore follows $\ker \rho$ lies in all Borel subgroups of G. Theorem 3.2.2.9 says that $W(G,T) \cong W(G',T')$ and $\mathscr{B}^T \cong \mathscr{B}'^{T'}$, where $T' = \rho(T)$ is a maximal torus of G'. These isomorphisms imply that if we can prove the Proposition for the group G', then it is immediately true for G, too. We can therefore assume that $G \subset GL(V)$.

Indeed, if we choose a basis of V consisting of weight vectors relative to T then we can view elements of G as matrices in $GL(n, K)$, say. In this case, T consists of diagonal matrices. Now, $TU = B$, and so $TU_\alpha \subset B$, hence TU_α is closed, connected and solvable. The Lie-Kolchin Theorem implies that this group is conjugate to a group consisting of upper triangular matrices, so without loss of generality we may take U_α to consist of upper triangular matrices with 1's on the diagonal.

Now, Theorem 3.4.5.1 (3) says that there exists an isomorphism

$$\epsilon_\alpha : \mathbb{G}_a \to U_\alpha$$

such that

$$t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x). \tag{3.24}$$

Since ϵ_α is a morphism of algebraic varieties, $\epsilon_\alpha(x)$ must be a polynomial in $x \in \mathbb{G}_a$, and, indeed, the $(i,j)^{\text{th}}$ entry is polynomial in x. That is, for some $m \in \mathbb{N}$,

$$\epsilon_\alpha(x)_{ij} = c_0 + c_1 x + \cdots + c_m x^m \tag{3.25}$$

where $c_i \in K$ is independent of x. Now $\alpha(t)x \in \mathbb{G}_a$, too, so

$$\epsilon_\alpha(\alpha(t)x)_{ij} = c_0 + c_1(\alpha(t)x) + \cdots + c_m(\alpha(t)x)^m \tag{3.26}$$

On the other hand, by direct calculation, we have $(t\epsilon_\alpha(x)t^{-1})_{ij} = (t_i/t_j)\epsilon_\alpha(x)_{ij}$, where $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$. Combining this with equations (3.24) and (3.26) gives

$$c_0\big(1 - t_i/t_j\big) + c_1\big(\alpha(t) - t_i/t_j\big)x + \cdots + c_m\big(\alpha(t)^m - t_i/t_j\big)x^m = 0 \tag{3.27}$$

Now this is a polynomial in x with coefficients in K, and so each of these coefficients must vanish. Therefore, if $c_k \neq 0$, then $\alpha(t)^k = t_i/t_j$. Since α is a root of G, it is not a trivial character, and therefore distinct powers of α must themselves be distinct. In particular, there is only one k for which $\alpha(t)^k = t_i/t_j$, and so all $c_l = 0$ if $l \neq k$. Combining this with equation (3.25) gives

$$\epsilon_\alpha(x)_{ij} = c_k x^k$$

Suppose the j^{th} basis vector e_j of V is of weight χ, that is, $\chi(t) = t_j$. If $u \in U_\alpha$, then ue_j is the j^{th} column of u. We just saw above that the $(i,j)^{\text{th}}$ entry of an element u is of the form $c_k x^k$ for some $x \in \mathbb{G}_a$. If this entry is nonzero, then $c_k \neq 0$, which we saw means $\alpha^k(t) = t_i/t_j$, for k a non-negative integer. If μ is the weight of the i^{th} basis vector, this means $\alpha^k = \chi\nu^{-1}$, or $\nu = \chi + k\alpha$ in additive notation. Note also that the j^{th} coordinate of ue_j is always one, since u is upper triangular with 1's on the diagonal.

□

Corollary 3.5.1.4. *Let $\alpha, \beta \in \Phi$. Then* $\mathrm{Ad}Z_\alpha$ *stabilises the subspace* $\mathfrak{m} = \sum \mathfrak{g}_{\beta+k\alpha} \subset \mathfrak{g}$ *where $k \in \mathbb{Z}$.*

Proof. By Theorem 3.4.3.2(3), the roots of Z_α are $\pm\alpha$, and so, applying Theorem 3.4.5.1 (4) gives us that Z_α is generated by the groups T, U_α and $U_{-\alpha}$. It therefore suffices to show that $\mathfrak{m}$ is stabilised by $\mathrm{Ad}T, \mathrm{Ad}U_\alpha$ and $\mathrm{Ad}U_{-\alpha}$. If $t \in T$, then, for $k \in \mathbb{Z}$, $\mathsf{x} \in \mathfrak{g}_{\beta+k\alpha}$,

$$\mathrm{Ad}t(\mathsf{x}) = t\mathsf{x}t^{-1} = (\beta + k\alpha)(t)\mathsf{x}$$

and so $\mathrm{Ad}T$ certainly stabilises $\mathfrak{m}$.

We now wish to show that U_α stabilises $\mathfrak{m}$. For this we apply Proposition 3.5.1.3 to the representation

$$\mathrm{Ad} : Z_\alpha \to GL(\mathfrak{z}_\alpha)$$

Lemma 3.5.1.2 assures us that the preconditions of the Proposition are met, and so, for any root $\lambda \in \Phi$, we have $\mathrm{Ad}U_\alpha(\mathfrak{g}_\lambda) \subset \sum \mathfrak{g}_{\lambda+k\alpha}$ for $k \in \mathbb{Z}^+$. Therefore, if $\mathsf{x} \in \mathfrak{g}_{\beta+k'\alpha}$, for some $k' \in \mathbb{Z}$, setting $\beta + k'\alpha = \lambda$ give us

$$(\mathrm{Ad}U_\alpha)\mathsf{x} \subset \sum \mathfrak{g}_{\lambda+k'\alpha} = \sum \mathfrak{g}_{\beta+(k+k')\alpha} \subset \mathfrak{m}$$

and therefore $\mathrm{Ad}U_\alpha$ stabilises $\mathfrak{m}$ as required. Note that another application of Proposition 3.5.1.3 equally well says that, for $\lambda \in \Phi$, we have $\mathrm{Ad}U_{-\alpha}(\mathfrak{g}_\lambda) \subset \sum \mathfrak{g}_{\lambda+k(-\alpha)}$. Therefore, if $\mathsf{x} \in \mathfrak{g}_{\beta+k'\alpha}$, setting $\lambda \in \beta + k'\alpha$ gives

$$(\mathrm{Ad}U_{-\alpha})\mathsf{x} \subset \sum \mathfrak{g}_{\lambda+k(-\alpha)} = \sum \mathfrak{g}_{\lambda+(k'-k)\alpha} \subset \mathfrak{m}$$

and so $\mathrm{Ad}U_{-\alpha}$ stabilises $\mathfrak{m}$. Putting this all together shows us that Z_α stabilises $\mathfrak{m}$, as required. □

Theorem 3.5.1.5. *Let G be semisimple, and set $E = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}$. Then Φ is an abstract root system in E, whose rank is* $\mathrm{rank}(G)$, *and whose abstract Weyl group $W(\Phi)$ is isomorphic to W.*

Proof. We need to verify the axioms (1)-(4) of Definition 3.5.1.1.

Beginning with Definition 3.5.1.1(1), note that since $\mathrm{Card}(\Phi) = \dim G - \mathrm{rank}(G)$ by Theorem 3.4.3.2, it is clear that Φ is finite. By definition, Φ does not contain 0, and, by construction of E, $\dim E = \mathrm{rank}(\mathrm{X}(T)) = \mathrm{rank}(G)$.

For Definition 3.5.1.1(2), suppose $\alpha \in \Phi$, and also $k\alpha \in \Phi$, for some scalar k. We claim that this implies $k\alpha$ is also a root of Z_α. For, suppose $0 \neq \mathsf{x} \in \mathfrak{g}_{k\alpha}$, and $t \in T_\alpha = (\ker \alpha)^\circ$. Then

$$t\mathsf{x}t^{-1} = k\alpha(t)\mathsf{x} = \mathsf{x}$$

and so $\mathsf{x} \in \mathfrak{c}_\mathfrak{g}(T_\alpha) = \mathscr{L}(C_G(T_\alpha)) = \mathscr{L}(Z_\alpha)$. It follows that $k\alpha$ is a root of Z_α. But we know from Theorem 3.4.3.2(3) that the root space decomposition of Z_α is

$$\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

and so the only multiples of α are $\pm\alpha$, as required.

For Definition 3.5.1.1(3), we use Proposition 3.3.1.5. Since W_α is of order 2, there exists a nontrivial element $\sigma_\alpha \in W_\alpha \subset W$ where the inclusion is by Lemma 3.3.1.7. Now σ_α is non trivial and permutes the roots of Z_α, which we just argued are precisely $\pm\alpha$. It follows, therefore, that σ_α sends α to $-\alpha$. In terms of geometry, a reflection is an invertible linear transformation which leaves some hyperplane fixed, and sends any element orthogonal to that hyperplane to its negative. We therefore need to show that σ_α fixes some hyperplane in $E = \mathbb{R} \otimes_\mathbb{Z} \mathrm{X}(T)$, which is to say, σ_α fixes a subgroup of $\mathrm{X}(T)$ of rank 1 less than that of $\mathrm{X}(T)$.

Choose a 1 parameter subgroup $\lambda \in \mathrm{Y}(T)$ such that $\sigma_\alpha(\lambda) = -\lambda$. That is, if $n \in N_G(T)$ is a representative for σ_α, then

$$n\lambda(a)n^{-1} = -\lambda(a)$$

for all $a \in \mathbb{G}_m$. Define a subset $V \subset W$ as follows:

$$V = \{\chi \in \mathrm{X}(T) \mid \langle \chi, \lambda \rangle = 0\}$$

Then V is a subgroup, whose rank is one less than that of $\mathrm{X}(T)$. Consider now the set of commutators

$$(n, T) \subset \{ntn^{-1}t^{-1} \mid t \in T\}$$

It is easy to verify that this is in fact a subtorus of T, and its image is nontrivial in T/T_α, the latter being a torus of dimension 1, which means (n, T) is also of dimension 1. Evidently $\mathrm{im}\lambda \subset (n, T)$, and since they are of equal dimension, they must be equal. Now, if $\chi \in V$, then $\mathrm{im}\lambda = (n, T) \subset \ker \chi$, which means $\chi((n, T)) = \{e\}$, which means $\chi(ntn^{-1}) = \chi(t)$ for all t, and therefore σ fixes χ. Therefore σ is a reflection, as required by Definition 3.5.1.1(3).

For Definition 3.5.1.1(4), apply Corollary 3.5.1.4. Indeed, the W-action on Φ sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{\sigma\alpha} = \mathrm{Ad}n(\mathfrak{g}_\alpha)$, by Proposition 3.2.3.3, where n is some representative of σ. Since we decided above that σ_α is the nontrivial reflection in the Weyl group of Z_α, there is a representative n_α for σ_α which lies in $N_{Z_\alpha}(T)$. Coroallry 3.5.1.4 says $\mathrm{Ad}n_\alpha$ stabilises $\sum \mathfrak{g}_{\beta+k\alpha}$, and in particular, sends an element $\mathsf{x} \in \mathfrak{g}_\beta$ to some element $\mathrm{Ad}n_\alpha \mathsf{x}$ to the root space $\mathfrak{g}_{\beta+k\alpha}$ for some integer k. Since G is reductive, both $\mathfrak{g}_\beta$ and $\mathfrak{g}_{\beta+k\alpha}$ are of dimension one, and so it follows that $\mathrm{Ad}n(\mathfrak{g}_\beta) = \mathfrak{g}_{\beta+k\alpha}$, which in turn means $\sigma_\alpha(\beta) = \beta + k\alpha$, and so Definition 3.5.1.1(4) holds. □

Corollary 3.5.1.6. *Let G be reductive, with maximal torus T. Then $\Phi = \Phi(G,T)$ is an abstract root system.*

Proof. Let $G' = G/Z(G)^\circ$, where $Z(G)^\circ = R(G)$ by Lemma 3.1.1.9. Thus both G and G' are reductive, so Theorem 3.4.3.2 applies to both groups. Since $Z(G)^\circ \subset C_G(T)$, the roots of G coincide with those of G'. Moreover, since $R(G)$ lies in every Borel subgroup of G, by Lemma 3.1.7.2, Proposition 3.2.2.9 tells us that $W'(G',T') \cong W(G,T)$, where T' is the image of T in G'. Since root systems are evidently determined by the roots and the corresonding reflections, it follows that $\Phi(G,T)$ is an abstract root system which coincides with $\Phi(G',T')$. □

3.5.2 Further Properties of the Root System

Now that we know Φ is a root system, we can exploit all the known facts about root systems to help us understand the structure of G. The facts that we are interested in are set out in detail in §9 and 10 of [5]. In the first place, we wish to know something about the *bases* of our root system. Recall that a base Δ is a subset $\{\alpha_1, \dots, \alpha_l\}$ of linearly independent roots which span Φ, and such that each element $\alpha \in \Phi$ can be expressed in the form $\sum c_i\alpha_i$ for c_i integers which are all of the same sign.

Proposition 3.5.2.1. *Let B be a Borel subgroup of G which contains a maximal torus T. Then B determines a partition of Φ into $\Phi(B) \cup -\Phi(B)$. This partition, moreover, determines a base $\Delta(B)$ of Φ.*

Proof. We have seen already that, for each $\alpha \in \Phi$, then $\Phi(Z_\alpha, T) \cap \Phi = \{\alpha, -\alpha\}$. Moreover, the Borel subgroup $B \cap Z_\alpha$ contains precisely one of these roots. By intersecting B with all such groups Z_α, we get a collection $\Phi(B)$ of exactly half of the roots of G. Furthermore, if $\alpha \in \Phi(B)$, then $-\alpha \notin \Phi(B)$. Since $\Phi = -\Phi$, it therefore follows that $\Phi = \Phi(B) \cup -\Phi(B)$.

We define a subset $\Delta(B)$ of $\Phi(B)$ as follows: $\alpha \in \Delta(B)$ if it cannot be expressed as the sum of two elements $\beta_1, \beta_2 \in \Phi(B)$. The elements of $\Delta(B)$ are in this sense the 'minimal' roots of $\Phi(B)$. In fact, it is a standard theorem of abstract root systems that a construction such as this gives a basis; see, for example, Theorem 10.1 of [5]. In particular, each element of $\Phi(B)$ can be written as a non-negative integer combination of elements of $\Delta(B)$, and therefore the elements of $\Delta(B)$ span Φ. Thus $\Phi(B)$ consists of the *positive roots* with respect to this base $\Delta(B)$. □

Remark. Proposition 3.5.2.1 is really a consequence of the fact that each Borel subgroup $B \in \mathscr{B}^T$ defines a unique *Weyl chamber*, namely a subset $\mathfrak{C}(B)$ of $\mathrm{Y}(T)$ consisting of 1 parameter subgroups which are firstly *regular*, that is, such that $\mathrm{im}\lambda$ is a regular torus, and are secondly such that $\langle \alpha, \lambda \rangle > 0$ if and only if α is a root of B. It is an immediate fact that a regular 1 parameter subgroup λ has the property that $\langle \beta, \lambda \rangle \neq 0$ for all $\beta \in \Psi$, so each element of $\mathfrak{C}(B)$ partitions Φ in precisely the fashion described in Proposition 3.5.2.1. We omit the proofs here, but it can be shown that this notion of Weyl chamber corresponds with the notion of Weyl chamber for abstract groups.

Proposition 3.5.2.2. *Let G be a reductive group with maximal torus T, and let Δ be a base for its root system $\Phi(G,T)$. Then $\Delta = \Delta(B)$ for some Borel subgroup $B \in \mathscr{B}^T$ of G.*

Proof. By definition, a base Δ defines a partition of Φ into positive roots Φ^+ and negative roots, Φ^-, such that $-\Phi^+ = \Phi^-$. It follows that $\mathrm{Card}(\Phi^+) = \frac{1}{2}\mathrm{Card}(\Phi)$.

On the other hand, it follows from Corollary 3.4.5.2 that every Borel subgroup B has exactly $\frac{1}{2}\mathrm{Card}(\Phi)$ distinct roots, and, moreover, none of these roots are multiples of each other. In particular, if α is a root of B, then $-\alpha$ is not. The groups T and U_α with $\alpha \in \Phi^+$ therefore generate a group H of dimension $\mathrm{rank}(G) + \frac{1}{2}\mathrm{Card}(\Phi)$, and which has exactly $\frac{1}{2}\mathrm{Card}(\Phi)$ distinct roots, none of which are multiples of each other. Therefore $\mathfrak{h} \subset \mathscr{L}(B)$ for some $B \in \mathscr{B}^T$, and since the dimensions are equal, it follows that these groups are equal. □

Combining Propositions 3.5.2.1 and 3.5.2.2 gives:

Corollary 3.5.2.3. *For a reductive group G with maximal torus T, the Borel subgroups containing T are in 1-1 correspondence with the set of bases of $\Phi(G,T)$.*

Finally we include a lemma which will help us to identify the simple reflections in Chapter 5:

Lemma 3.5.2.4. *Let Δ be a base, with associated positive roots Φ^+. Suppose $\sigma \in W$ is such that $\sigma(\alpha) = -\alpha$ for some $\alpha \in \Delta$, and furthermore $\sigma(\beta) \in \Phi^+ - \alpha$ for all $\beta \in \Delta - \{\alpha\}$. Then σ is equal to the reflection σ_α.*

Proof. It is a standard fact of abstract root systems that W is generated by the simple reflections, that is, those reflections associated to the elements of Δ; see, for example, Theorem 10.3 of [5]. For each element $\tau \in W$, then, we can express τ as a word consisting of simple reflections. We define the *length* $l(\tau)$ of τ to be the minimal length of such a word. We also wish to assign to each $\tau \in W$ the integer $n(\tau)$, which is defined to be the number of positive roots which τ sends to Φ^-.

The result we need here is Lemma 10.3A of [5], which says that, for all $\tau \in W$, $l(\tau) = n(\tau)$. We can therefore immediately apply this result to the hypotheses we have here, to see that $n(\sigma) = 1$. Indeed, we write $\Delta = \{\alpha_1, \dots, \alpha_m\}$ and set $\alpha = \alpha_1$, so that we can express any positive root $\beta \in \Phi^+$ in the form $\beta = \sum c_i\alpha_i$ for $c_i \in \mathbb{Z}^+$. Then, if $\alpha \neq \beta$, we have $c_j \neq 0$ for some $j \neq 1$. Thus $\sigma(\beta) = \sum c_i\sigma(\alpha_i)$ and so, by our hypothesis, the coefficient of α_j in $\sigma(\beta)$ is positive, which has two consequences. Firstly, it means $\sigma(\beta) \neq \alpha$, and secondly, since Δ is a base, all the coefficients of $\sigma(\beta)$ are of the same sign, and therefore in this case they are all non negative. Therefore $\sigma(\beta) \in \Phi^+ - \{\alpha\}$, or, equivalently, σ permutes the set $\Phi^+ - \{\alpha\}$. On the other hand, $\sigma(\alpha) = -\alpha$, and so $n(\sigma) = 1$.

Therefore σ has length 1, which is to say σ is a simple reflection. Since $\sigma(\alpha) = -\alpha$, it follows that $\sigma = \sigma_\alpha$ as required. □

3.6 Bruhat Decomposition

We conclude this chapter with another way to approach the structure of reductive groups, namely by giving them a normal form which is parametrised by Borel subgroups and elements of the Weyl group. This is the famous Bruhat decomposition.

3.6.1 Set Up

Throughout this section, G will be a reductive group, with a maximal torus T, and B a fixed element of $\mathscr{B}^T$. By Corollary 3.5.2.3, this amounts to a choice of a fixed base $\Delta = \Delta(B)$ of $\Phi = \Phi(G,T)$. We will also use the notation $U = B_u$, and $N = N_G(T)$. We begin with a lemma:

Lemma 3.6.1.1. *The root space decomposition of U is given by*

$$\mathscr{L}(U) = \mathfrak{u} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$$

Proof. Recall, $\mathscr{L}(B) = \mathfrak{t}\oplus\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. However, since $B = TU = UT$, it follows that $\mathscr{L}(B) = \mathfrak{t}\oplus\mathfrak{u}$, whence $\mathfrak{u} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. □

Definition 3.6.1.2. An algebraic group H is *directly spanned* by its closed subgroups $H_1, \dots, H_n$ in the given order, if the product morphism

$$H_1 \times \cdots \times H_n \longrightarrow H$$

is bijective.

Lemma 3.6.1.3. *The T-action on U fixes only the element the identity, that is, $U^T = \{e\}$.*

Proof. Suppose $g \in U$ is such that $tgt^{-1} = g$ for all $t \in T$. Then $g \in C_G(T) = T$, where the equality is by Corollary 3.4.2.1(2). But $T \cap U = \{e\}$, as required. □

Proposition 3.6.1.4. *Let H be a closed, T-stable subgroup of U. Then H is connected and is directly spanned by those U_α's, taken in any order, for which $\mathscr{L}(U) = \mathfrak{g}_\alpha$ lies in $\mathfrak{h}$.*

Proof. Let $\Phi' = \{\alpha_1, \dots, \alpha_n\}$ be the roots of H. By Proposition 2.3.1.9, $U_{\alpha_i} = C_U(T_{\alpha_i}) \subset H$ for each i, so we can define a product morphism

$$\pi : U_1 \times \cdots U_n \longrightarrow H$$

We need to show that π is an isomorphism.

Suppose firstly that H is connected. If H is also commutative, then π is a homomorphism, and $\ker\pi$ is a finite, T-stable subgroup. Since algebraic group actions stabilise connected components, by Proposition 1.4.1.7(4), it follows that $\ker\pi \subset H^T = \{e\}$. Since H is assumed to be connected and $\dim \mathrm{im}\pi = \dim H$, we have $\mathrm{im}\pi = H$, and so π is an isomorphism.

We can now drop the assumption that H is commutative, but apply the above arguments to $Z = Z(H)^{\circ}$, which, by Proposition 2.2.3.6(1) is of positive dimension, since H is nilpotent, being a subgroup of U. So Z is directly spanned by the U_{α_i}'s it contains. Suppose these are labelled $U_{\alpha_{l+1}}, \dots, U_{\alpha_n}$. Then if $f : H \to H/Z$ is the quotient map, T acts on H/Z, which is of dimension less than H, and so by induction H/Z is directly spanned by $\varphi(U_{\alpha_1}), \dots, \varphi(U_{\alpha_l})$, and so H is directly spanned by $U_{\alpha_1}, \dots U_{\alpha_n}$.

We can now drop the assumption that H is connected. Indeed, since we now know that U and H° are directly spanned by the relevant U_α's, we can write $U = H^{\circ}V$, where V is the product of those U_α's such that α is not a root of H°. Now $H \cap V$ is finite and T-stable, and so another application of Proposition 1.4.1.7(4) tells us that T acts trivially on $H \cap V$, and therefore must again be equal to $\{e\}$. The result is now proven. $\square$

Now Proposition 3.6.1.4 does not say which subsets of Φ^+ give rise to T-stable subgroups of U.

Lemma 3.6.1.5. *Let $\sigma \in W$ be represented by $n \in N$. Then nUn^{-1} is independent of the choice of n.*

Proof. It is a simple consequence of Theorem 3.1.7.3 that $N_G(U) = B$. Since $T \subset B$, it follows that T normalises U. Suppose, then, that $\sigma = nT = mT$ for $m, n \in T$. Then $m^{-1}n \in T$, so $(m^{-1}n)U(m^{-1}n)^{-1} = U$, whence $nUn^{-1} = mUm^{-1}$, as required. $\square$

Lemma 3.6.1.5 tells us that we can write $\sigma U \sigma^{-1}$ without fear of confusion. Recall also that $nU_\alpha n^{-1} = U_{\sigma(\alpha)}$ for $n \in N$ representing σ, whence we right $\sigma U_\alpha \sigma^{-1} = U_{\sigma(\alpha)}$.

Let $B^- = TU^-$ be opposite to B. That is, $B \cap B^- = T$, and U^- is generated by the groups $U_{-\alpha}$ for $\alpha \in \Phi^+$. Define groups

$$U_\sigma = U \cap \sigma U \sigma^{-1} \text{ and} \tag{3.28}$$

$$U'_\sigma = U \cap \sigma U^- \sigma^{-1} \tag{3.29}$$

Proposition 3.6.1.6. *The groups U_σ, U'_σ are T-stable subgroups of U, with respective sets of roots*

$$\Phi^+_\sigma = \{\alpha > 0 \,|\, \sigma(\alpha) > 0\} \tag{3.30}$$

$$\Phi^-_\sigma = \{\alpha > 0 \,|\, \sigma(\alpha) < 0\} \tag{3.31}$$

Furthermore, the sets Φ^+_σ and Φ^-_σ partition Φ^+.

Proof. Suppose $\alpha \in \Phi^+_\sigma$. Then $U_\alpha \subset U$ and $U_\alpha \subset \sigma U \sigma^{-1}$. But this second expression amounts to the following:

$$U_{\sigma^{-1}\alpha} = \sigma^{-1} U_\alpha \sigma \subset U$$

where the first equality is by Theorem 3.4.5.1 (2), and this expression implies that $\sigma^{-1}\alpha$ is a positive root . We have shown that $\Phi^+_\sigma \subset \{\alpha > 0 \,|\, \sigma(\alpha) > 0\}$. On the other hand, if α is a positive root such that $\sigma\alpha$ is positive, then $U_\alpha \subset U$, and $U_{\sigma^{-1}\alpha} = \sigma^{-1}U_\alpha\sigma \subset U$, and together these expressions imply $U_\alpha \subset \sigma U\sigma^{-1} \cap U$, whence $\alpha \in \Phi_\sigma^+$. We have shown that $\Phi_\sigma^+ = \{\alpha > 0 \mid \sigma(\alpha) > 0\}$. The expression for Φ_σ^- follows from a similar argument.

Finally, suppose $\alpha \in \Phi^+$. Then $\sigma(\alpha)$ is certainly a root, and so is either positive root or negative. It follows that $\Phi^+ = \Phi_\sigma^+ \cup \Phi_\sigma^-$, a disjoint union, as required. □

By Proposition 3.6.1.6, combined with Proposition 3.6.1.4 we have, for any $\sigma \in W$,

$$U = U_\sigma U'_\sigma = U'_\sigma U_\sigma \tag{3.32}$$

Let $\alpha \in \Delta$, and write $\sigma = \sigma_\alpha$.

Corollary 3.6.1.7. *U_{σ_α} has codimension 1 in U. Moreover, $U = U_\alpha U_{\sigma_\alpha}$.*

Proof. Obviously, $\sigma_\alpha(\alpha) = -\alpha$, and so, by Proposition 3.6.1.6, $\alpha \notin \Phi_\sigma^+$. On the other hand, Lemma 10.2B of [5] tells us that σ permutes the roots of $\Phi^+ - \{\alpha\}$, and so any positive root β distinct from α has $\sigma(\beta) \in \Phi^+$, and so, by Proposition 3.6.1.6 again, $\beta \in \Phi_\sigma^+$. It follows that

$$\mathscr{L}(U_\sigma) = \bigoplus_{\beta \in \Phi^+ - \{\alpha\}} \mathfrak{g}_\beta$$

and in particular, $\dim U_\sigma = \dim U - 1$ and $U = U_\alpha U_{\sigma_\alpha}$, as required. □

We should also note here that $U^- = U_{-\alpha}U_{\sigma_{-\alpha}} = U_{-\alpha}U_{\sigma_\alpha}$.

Corollary 3.6.1.8. *U_{σ_α} is normal in U.*

Proof. Corollary 3.6.1.7 allows us to apply Proposition 2.2.3.6(2), with $U = G$ and $U_{\sigma_\alpha} = H$, and the result is immediate. □

Proposition 3.6.1.9. *$Z_\alpha \subset N_G(U_{\sigma_\alpha})$. In particular, $Z_\alpha U_{\sigma_\alpha} = U_{\sigma_\alpha} Z_\alpha$.*

Proof. In the first place, note that

$$\sigma_\alpha(U_{\sigma_\alpha})\sigma_\alpha^{-1} = \sigma_\alpha U \sigma_\alpha^{-1} \cap \sigma_\alpha(\sigma_\alpha U \sigma_\alpha^{-1})\sigma_\alpha^{-1} = U_{\sigma_\alpha}$$

where the last equality is since $\sigma_\alpha^2 = 1$. Since the Weyl group of Z_α is of order 2, it follows that $n \in N_{Z_\alpha}(T) - T$ is a representative for σ_α, and so $N_{Z_\alpha}(T) \subset N_G(U_{\sigma_\alpha})$.

Therefore, $TU_{\sigma_\alpha} = U_{\sigma_\alpha}T$, and we know from above that $U_\alpha U_{\sigma_\alpha} = U_{\sigma_\alpha}U_\alpha$, and similarly for $U_{-\alpha}$. Since Z_α is generated by T, U_α and $U_{-\alpha}$, it follows that Z_α normalises U_{σ_α}. □

Lemma 3.6.1.10. *Let $\sigma \in W$, with $n, m \in N$ both coset representatives for σ. Then $BnB = BmB$, and therefore is independent of the choice of coset representative.*

Proof. Suppose $\sigma = nT = mT$, then $m^{-1}n \in T$. Certainly $T \subset B$, which implies $m^{-1}nB = B$, or $nB = mB$. Finally, multiplying both sides on the right by B, we get $BnB = BmB$ as required. □

Lemma 3.6.1.10 allows us to write $B\sigma B$ without fear of ambiguity.

Lemma 3.6.1.11. $B\sigma B = U\sigma B = U\sigma TU$

Proof. Let $\sigma = nT$ for a fixed $n \in N$. Recall, firstly, that $B = UT = TU$. Then, on the one hand,

$$B\sigma B = BnB = UTnB = UnTB = U\sigma B$$

Similarly,

$$B\sigma B = BnB = UTnTU$$

Now, $n \in N_G(T)$, so $T = nTn^{-1}$, and therefore

$$B\sigma B = UTnTU = U(nTn^{-1})nTU = U(nT)TU = U\sigma TU$$

□

3.6.2 Semisimple Rank 1 Results

Lemma 3.6.2.1. *Suppose G is reductive, and $\varphi : G \to G'$ is an epimorphism, with* $\ker\varphi \subset Z(G)$. *Then G' is reductive.*

Proof. Evidently $\ker\varphi$ is solvable, by assumption, and since it is also normal, we have $\ker\varphi^\circ \subset R(G)$. But Lemma 3.1.7.2 then implies that $\ker\varphi^\circ \subset B$ for all Borel subgroups B of G.

We wish to show that $\varphi(\bigcap B) = \bigcap \varphi(B)$, since by Corollary 3.1.4.7, the latter intersection exhausts all of the Borel subgroups of G'. Certainly $\varphi(\bigcap B) \subset \bigcap \varphi(B)$. On the other hand, suppose $y \in \varphi(B) \cap \varphi(B')$ for two Borel subgroups B, B' of G. Then $y = \varphi(a) = \varphi(b)$ where $a \in B, b \in B'$. Therefore $ab^{-1} \in \ker\varphi \subset B'$, so $a \in B \cap B'$, and therefore $y \in \varphi(B \cap B')$. But this holds for all Borel subgroups, and so $\bigcap \varphi(B) \subset \varphi(\bigcap B)$. Therefore, $\bigcap(\varphi(B))^\circ = \varphi(\bigcap(B)^\circ$, which by Lemma 3.1.7.2 says $R(G') = \varphi(R(G))$.

In particular, $R_u(G') = \varphi(R(G))_u = \varphi(R_u(G))$, and so the fact that G is reductive implies that G' is, too. □

Lemma 3.6.2.2. *Let $\varphi : G \to G'$ be an epimorphism with* $\ker\varphi \subset Z(G)$. *Then, setting Borel subgroup $B' = \varphi(B) \in \mathscr{B}^{T'}$, maximal torus $T' = \varphi(T)$, and normaliser $N' = \varphi(N)$,*

$$G' = B'N'B' \iff G = BNB$$

Proof. ($\Longrightarrow$): Let $g \in G$. Then $\varphi(g) \in G' = B'N'B'$ and so $\varphi(g) = a'n'b'$ for $a', b' \in B'$ and $n' \in N'$. But $B' = \varphi(B), N' = \varphi(N)$, and so

$$\varphi(g) = a'n'b' = \varphi(a)\varphi(n)\varphi(b) = \varphi(anb)$$

for $a, b \in B, n \in N$. Then $g^{-1}anb \in Z(G) \subset T$, where the inclusion is since G is reductive. Therefore $g^{-1}anb = c$ for some $c \in T \subset B$. Rewriting this, we get $g = anbc^{-1}$, but $bc \in B$, so this implies $g \in BNB$, but g was arbitrary, so $G \subset BNB$. On the other hand, certainly $BNB \subset G$, so $G = BNB$.

($\Longleftarrow$): Let $g' \in G'$, that is, $g' = \varphi(g)$ for some $g \in G$. But $g = anb$ for some $a, b \in B, n \in N$. Then

$$g' = \varphi(g) = \varphi(anb) = \varphi(a)\varphi(n)\varphi(b) = a'n'b'$$

for $a', b' \in B', n' \in N'$, and so $G' \subset B'N'B'$. On the other hand, certainly $B'N'B' \subset G'$, so $G' = B'N'B'$ as required. □

Proposition 3.6.2.3. *Let* $\mathrm{rank}_{ss}(G) = 1$. *Then* $G = B \dot{\cup} B\sigma B$ *where* σ *is the nontrivial element in* W. *Moreover,* $B\sigma B = U\sigma TU$, *and, for* n *a fixed representative of* σ, *each element of* $B\sigma B$ *has a unique expression of the form* $untu'$, *for* $u, u' \in U$ *and* $t \in T$.

Proof. We first demonstrate that, for G of semisimple rank 1, $BNB = B \cup B\sigma B$. Suppose $g \in B \cup B\sigma B$. Then either $g \in B \subset BNB$, or $g = anb$ for $a, b \in B$, and $n \in N$ representing σ. Then $g \in BNB$, too, so $B \cup B\sigma B \subset BNB$. On the other hand, suppose $g \in BNB$, that is, $g = anb$, for $a, b \in B$ and $n \in N$. Suppose firstly that $n \in T \subset B$, then $g \in B$. On the other hand, suppose $n \notin T$. Then $nT \in W$ is nontrivial, and so must equal σ, since W is of order 2. Then $g \in B\sigma B$, and so $BNB \subset B \cup B\sigma B$ as required.

Suppose $c \in B \cap B\sigma B$. Then $c = anb$ for $a, b \in B$, $n \in N$. In particular, $n = a^{-1}cb^{-1} \in B \cap N$. But we know that W acts simply transitively on $\mathscr{B}^T$, which implies that if $x \in N$ is such that $xBx^{-1} = B$, then $x \in C_G(T) = T$ (where the latter equality is since G is reductive). But since $n \in B$, we certainly have $nBn^{-1} = B$, hence $n \in T$, and so $\sigma = e$, which is contrary to the assumption of the theorem. Therefore $B \cap B\sigma B = \emptyset$, and so the union $B \cup B\sigma B$ is disjoint, as required.

We now prove the theorem for the case $G = SL(2, K)$, with $B = T(2, K) \cap SL(2, K)$. It suffices to show that any element $g \in G - B$ has a unique expression of the form $untu'$, as indicated. Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ does not lie in B, that is, $c \neq 0$. The we can rewrite g as

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{3.33}$$

Now suppose the fixed n is the theorem is $n = \begin{pmatrix} 0 & -x \\ 1/x & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} 0 & -x \\ 1/x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/x & 0 \\ 0 & x \end{pmatrix}$$

which implies

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x \\ 1/x & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \tag{3.34}$$

Combining (3.33) with (3.34) gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -x \\ 1/x & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} \tag{3.35}$$

which is a decomposition of the form $untu'$. Suppose now that $g = \tilde{u}n\tilde{t}\tilde{u}'$ is another decomposition, where $\tilde{u}, \tilde{u}' \in U, \tilde{t} \in T$. That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tilde{u}n\tilde{t}\tilde{u}' = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -x \\ 1/x & 0 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$$

Simplifying the right hand side, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ps/x & pqs/x - x/s \\ s/x & qs/x \end{pmatrix}$$

Then $s/x = c$, so $t = \tilde{t}$. Furthermore, this implies $a = ps/x = pc$, and so $p = a/c$, whence $u = \tilde{u}$. Finally, we have $d = qs/x = qc$, and so $q = d/c$, whence $u' = \tilde{u}'$, and so these results together show that the decomposition given in (3.35) is unique.

Recall that, since $\mathrm{rank}_{ss}(G) = 1$, Theorem 3.3.3.2 and Corollary 3.3.3.4 say there exists an epimorphism

$$\varphi : G \longrightarrow \mathrm{PGL}(2, K)$$

where $\ker \varphi = Z(G)$. By Lemma 3.6.2.2, then, it follows that $\mathrm{PGL}(2, K)$ satisfies the theorem, too. On the other hand, if G is an arbitrary group of semisimple rank 1, then by Theorem 3.3.3.2 and Corollary 3.3.3.4 again, there exists an epimorphism

$$\varphi : G \longrightarrow \mathrm{PGL}(2, K)$$

with $\ker \varphi = Z(G)$, and so another application of Lemma 3.6.2.2 tells us that, since the theorem is satisfied for $\mathrm{PGL}(2, K)$, it is satisfied for this arbitrary G, too, and we are done. $\square$

Corollary 3.6.2.4. *Let G be of semisimple rank 1, with $\sigma \in W$ the nontrivial element. Then U acts on G/B and*

$$U.B \cup U.\sigma B = G/B$$

Proof. It is clear that $U.B \cup U.\sigma B \subset G/B$, and so we need to prove the reverse inclusion. Let $gB \in G/B$. Since $G = B \cup B\sigma B$ by Proposition 3.6.2.3, we have either $g \in B$ or $g \in B\sigma B$. Suppose $g \in B = UT$. Then $g = ut$ for $u \in U, t \in T$, and so $gB = utB = uB \in U.B$. On the other hand, if $g \in B\sigma B$, then Proposition 3.6.2.3 again says $g = untu'$, and so $gB = untu'B$, but $B = TU$, and so $tu' \in B$, which implies $gB = unB \in U.\sigma B$, as required. $\square$

3.6.3 The Decomposition Theorem

Theorem 3.6.3.1. *$G = \dot{\bigcup}_{\sigma \in W} B\sigma B$, with $B\sigma B = B\tau B$ if and only if $\sigma = \tau$.*

Proof. Let $\alpha \in \Delta$, and consider the group Z_α. Then

$$\begin{aligned} Z_\alpha B\sigma B &= Z_\alpha U \sigma B && \text{by Lemma 3.6.1.11} \\ &= Z_\alpha U_{\sigma_\alpha} U_\alpha \sigma B && \text{by Corollary 3.6.1.7} \\ &= U_{\sigma_\alpha} Z_\alpha U_\alpha \sigma B && \text{by Proposition 3.6.1.9} \end{aligned}$$

But since $U_\alpha \subset Z_\alpha$, this becomes

$$Z_\alpha B\sigma B = U_{\sigma_\alpha} Z_\alpha \sigma B \tag{3.36}$$

Consider now the action of Z_α on the variety G/B, which is by left multiplication:

$$\begin{aligned} Z_\alpha \times G/B &\longrightarrow G/B \\ (z, gB) &\longmapsto zgB \end{aligned}$$

Consider an element $nB \in G/B$, for some $n \in W$, which we will also write as σB. If $z \in Z_\alpha \cap nBn^{-1}$, then $znB = nB$. On the other hand, if $z \in Z_\alpha$ such that $znB = nB$, then $z \in nBn^{-1}$, which tells us that the stabiliser of the element nB in Z_α is equal to $Z_\alpha \cap nBn^{-1}$. Corollary 3.1.6.8 tells us that this group is in fact a Borel subgroup of Z_α, since nBn^{-1} is a Borel subgroup of G. Now $T \subset Z_\alpha \cap nBn^{-1}$, but Proposition 3.3.1.5 tells us only two Borel subgroups of Z_α contain T. We will suppose that α is a root of $Z_\alpha \cap nBn^{-1}$ (which of course leaves open the possibility that $nBn^{-1} = B$) and denote $Z_\alpha \cap nBn^{-1}$ by B_α, while the other Borel subgroup containing T corresponds to $\sigma_\alpha B_\alpha$. Now B_α is a stabiliser, and so Z_α / B_α is in bijective correspondence with the orbit $Z_\alpha . nB$, which we write $Z_\alpha \sigma B$.

Now we apply Corollary 3.6.2.4 to the group Z_α. We know $(B_\alpha)_u = U_\alpha$, and σ_α is the nontrivial element of W_α, so the Corollary, combined with the above bijective correspondence, says

$$Z_\alpha \sigma B = Z_\alpha / B_\alpha = U_\alpha \sigma B \cup U_\alpha \sigma_\alpha \sigma B \tag{3.37}$$

Combining equation (3.37) with equation (3.36), we get

$$\begin{aligned} Z_\alpha B\sigma B &= U_{\sigma_\alpha}(Z_\alpha \sigma B) \\ &= U_{\sigma_\alpha}(U_\alpha \sigma B \cup U_\alpha \sigma_\alpha \sigma B) \\ &= U_{\sigma_\alpha} U_\alpha \sigma B \cup U_{\sigma_\alpha} U_\alpha \sigma_\alpha \sigma B \end{aligned}$$

Now, Corollary 3.6.1.7 tells us $U_{\sigma_\alpha} U_\alpha \sigma B = U\sigma B$, and in turn Lemma 3.6.1.11 tells us this is equal to $B\sigma B$. Similarly, Corollary 3.6.1.7 gives $U_{\sigma_\alpha} U_\alpha \sigma_\alpha \sigma B = U\sigma_\alpha \sigma B$, while Lemma 3.6.1.11 turns this into $B\sigma_\alpha \sigma B$. Putting this all together, we get

$$Z_\alpha B\sigma B = B\sigma B \cup B\sigma_\alpha \sigma B \tag{3.38}$$

which, in particular, gives

$$Z_\alpha BWB \subset BWB$$

Since G is reductive, it is generated by the subgroups Z_α, for $\alpha \in \Delta$. Moreover, Proposition 3.6.2.3 tells us that $Z_\alpha = BW_\alpha B \subset BWB$. $\square$

Corollary 3.6.3.2. *If $\alpha \in \Delta$ and $\sigma \in W$, then*

$$\sigma_\alpha B\sigma \subset B\sigma B \cup B\sigma_\alpha\sigma B$$

Proof. We use equation (3.38) from the proof of Theorem 3.6.3.1. In particular, since $\sigma_\alpha \in Z_\alpha$, we have

$$\sigma_\alpha B\sigma = \sigma_\alpha B\sigma e \subset Z_\alpha B\sigma B = B\sigma B \cup B\sigma_\alpha\sigma B$$

as required. $\square$

Corollary 3.6.3.3. *Let B' be any Borel subgroup of G. Then $B \cap B'$ includes a maximal torus of G.*

Proof. Since Borel subgroups are conjugate to each other, there exists an element $x \in G$ such that $xBx^{-1} = B'$. Theorem 3.6.3.1 says we can find elements $u \in U, b \in B$ and $\sigma \in W$ such that $g = u\sigma b$. Then

$$B' = u\sigma bB(u\sigma b)^{-1} = u\sigma B\sigma^{-1}u^{-1}$$

Set $T' = u\sigma T\sigma^{-1}u^{-1}$. Since T' is conjugate to T, it is a maximal tous. Certainly $T' \subset B'$. Now $\sigma T\sigma^{-1}$ really equals nTn^{-1}, for a representative $n \in N$, and so $\sigma T\sigma^{-1} = T$. Moreover, since $U \subset B$, we have $T \subset B = u^{-1}Bu$, and therefore $uTu^{-1} \subset B$, which implies $T' \subset B$, as required. $\square$

Theorem 3.6.3.4. *For each $\sigma \in W$, fix a coset representative $n(\sigma) \in N$. Then each element $x \in G$ can be written in the form $x = u'n(\sigma)tu$, where $\sigma \in W, t \in T, u \in U$ and $u' \in U'_\sigma$ are all uniquely determined by x.*

Proof. Note firstly that, for any $\sigma \in W$, equation (3.32) gives $U = U'_\sigma U_\sigma$. Therefore,

$$B\sigma B = U\sigma B = U'_\sigma U_\sigma \sigma B \tag{3.39}$$

Now $U_\sigma = U \cap \sigma U\sigma^{-1}$ by definition, which implies

$$\sigma^{-1}U_\sigma\sigma = \sigma^{-1}U\sigma \cap U \subset U$$

Therefore, $U_\sigma\sigma \subset \sigma U$, and plugging this into equation (3.39) gives

$$B\sigma B \subset U'_\sigma \sigma UB = U'_\sigma \sigma B \subset B\sigma B$$

and so $B\sigma B = U'_\sigma\sigma B = U'_\sigma\sigma TU$. Therefore, any element of $x \in G$ can be written in the required form $x = u'\sigma tu = u'n(\sigma)tu$. It remains to prove uniqueness of this expression.

Suppose $x = u'n(\sigma)tu = v'n(\sigma)sv$, for $u', v' \in U'_\sigma, s, t \in T, u, v \in U$. Then

$$tu = (n(\sigma))^{-1}u'^{-1}v'n(\sigma)sv \tag{3.40}$$

The expression $(n(\sigma))^{-1}u'^{-1}v'n(\sigma)$ lies in $\sigma^{-1}B\sigma = B$. On the other hand, since $u'^{-1}v' \in U'_\sigma$, we have $(n(\sigma))^{-1}u'^{-1}v'n(\sigma) \in \sigma^{-1}U'_\sigma\sigma$. Now $U'_\sigma = U \cap \sigma U^{-1}\sigma^{-1}$ by definition, which implies

$$\sigma^{-1}U_\sigma\sigma = \sigma^{-1}U^{-1}\sigma \cap U \subset U^{-1}$$

Therefore, $U_\sigma\sigma \subset \sigma U^{-1}$, and in particular, the expression $(n(\sigma))^{-1}u'^{-1}v'n(\sigma)$ lies in U^{-1}. This expression therefore lies in $B \cap U^{-1} = \{e\}$, which implies $v'n(\sigma) = u'n(\sigma)$, or $v' = u'$. Plugging this in to equation (3.40) gives $tu = sv$, or $s^{-1}t = vu^{-1} \in T \cap U$. But this intersection is necessarily trivial, and so $s = t$ and $v = u$, as required. $\square$

Chapter 4

Representations

This chapter offers a brief description of the representation of semisimple groups like the Classical Groups. The theory is not quite as simple as the analagous theory for semisimple Lie algebras, but nevertheless we will see that the irreducible representations can be parametrised by certain highest weights. The chapter concludes with an application to the canonical and unceasingly useful example of $\mathrm{SL}(2, K)$.

4.1 Weights

We assume G is a semisimple algebraic group. Since G is therefore reductive, we know from Theorem 3.4.3.2 that

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

and for each $\alpha \in \Phi$ there is a group Z_α of semisimple rank 1 such that

$$\mathscr{L}(Z_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

4.1.1 Weights of a Representation

Let $\rho : G \to GL(V)$ be a representation of G. Recall that $\chi \in \mathrm{X}(\rho(T))$ is a weight of $\rho(T)$ if the weight space

$$V_\chi = \{v \in V \mid y.v = \chi(y)v, \forall y \in \rho(T)\}$$

is nonzero. Of course, this weight space can be rewritten

$$V_\chi = \{v \in V \mid \rho(x).v = \chi(\rho(x))v, \forall x \in T\}$$

Note futher that the map

$$\begin{aligned} f : \mathrm{X}(\rho(T)) &\longrightarrow \mathrm{X}(T) \\ \chi &\longmapsto \chi \circ \rho \end{aligned}$$

is injective, and so, writing $\lambda = \chi \circ \rho \in \mathrm{X}(T)$, the above weight space can be denoted

$$V_\lambda = \{v \in V \mid \rho(x).v = \lambda(x)v, \forall x \in T\}$$

Definition 4.1.1.1. An element $\lambda \in \mathrm{X}(T)$ is a *weight* of ρ if it is of the form $\lambda = \chi \circ \rho$ for some weight $\chi \in \mathrm{X}(\rho(T))$ of $\rho(T)$. We call $\dim V_\lambda$ the *multiplicity* of λ, and we denote the set of weights of ρ by Λ.

It is an immediate consequence of Lemma 1.5.1.4 that that there are only finitely many weights of ρ.

Proposition 4.1.1.2. *W permutes the weights of ρ. Moreover, given a weight λ and an element, $\sigma \in W$, then* $\dim V_\lambda = \dim V_{\sigma(\lambda)}$.

Proof. Let $\lambda \in \Lambda$, and $\sigma \in W$, with $n \in N_G(T)$ a representative of σ. Then, for an arbitrary element $t \in T$, and non-zero vector $v \in V_\lambda$,

$$t(n.v) = n(n^{-1}tn).v = n\lambda(n^{-1}tn)v = \sigma(\lambda)(t)n.v$$

which means $n.v \in V_{\sigma(\lambda)}$, and so $\sigma(\lambda)$ is a weight of ρ.

Suppose now that $\{v_1, \ldots, v_m\}$ is a basis for V_λ. Then, if $a_1, \ldots, a_m$ are scalars such that

$$a_1(n.v_1) + \cdots + a_m(n.v_m) = 0,$$

it follows that

$$n^{-1}(a_1(n.v_1) + \cdots + a_m(n.v_m)) = a_1v_1 + \cdots + a_mv_m = 0$$

and so $a_i = 0$ for all i, which implies $\{n.v_1, \ldots, n.v_m\}$ is linearly independent. Likewise, suppose $w \in V_{\sigma(\lambda)}$. Then $n^{-1}.w \in V_\lambda$, and so $n^{-1}.w = \sum a_iv_i$ for some scalars a_i. Therefore, $w = \sum a_in.v_i$, and so $\{n.v_1, \ldots, n.v_m\}$ spans $V_{\sigma(\lambda)}$. It follows, then, that $\dim V_\lambda = \dim V_{\sigma(\lambda)}$, as required. □

Example 37. Let G be semisimple, and set ρ to be the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Then by Theorem 3.4.3.2, $\Lambda = \Phi \cup \{0\}$, that is, the nonzero weights of Ad are the roots of G, each with multiplicity 1, and the trivial weight 0, whose weight space corresponds to $\mathfrak{c}_{\mathfrak{g}}(T)$, and whose multiplicity is $\mathrm{rank}(G) = \dim T$.

4.1.2 Abstract Weights

Lemma 4.1.2.1. *Let $\rho : G \to GL(V)$ be a rational representation, and suppose further that $\alpha \in \Phi$. Then, for any weight $\lambda \in \Lambda$, $\rho(Z_\alpha)$ stabilises the subspace $\sum V_{\lambda+k\alpha}$ where $k \in \mathbb{Z}$.*

Proof. We saw in Proposition 3.5.1.3 that for $\alpha \in \Phi^+$, $\rho(U_\alpha)$ maps V_λ to $\sum V_{\lambda+k\alpha}$ for $k \in \mathbb{Z}^+$. By Theorem 3.4.5.1(4) applied to Z_α, we see that Z_α is generated by the groups T, U_α and $U_{-\alpha}$. Since T stabilises each weight space, it follows that $\rho(Z_\alpha)$ stabilises the subspace $\sum V_{\lambda+k\alpha}$ for $k \in \mathbb{Z}$. □

Proposition 4.1.2.2. *For a weight $\lambda \in \mathrm{X}(T)$ of $\rho : G \to GL(V)$ and a weight $\alpha \in \Phi$ of G, the reflection σ_α sends λ to $\lambda + k\alpha$ for some $k \in \mathbb{Z}$.*

Proof. This follows at once from Lemma 4.1.2.1. We saw in the proof of Proposition 4.1.1.2 that, given n representing an element $\sigma \in W$, then σ sends an element of $v \in V_\lambda$ to $\rho(n)v \in V_{\sigma(\lambda)}$. If $\sigma = \sigma_\alpha$ then $n = n_\alpha \in Z_\alpha$, and so Lemma 4.1.2.1 tells us that $\rho(n)v \in \sum V_{\lambda+k\alpha}$. Now we know from Lemma 1.5.1.4 that weight spaces are linearly independent, and so it follows that $\rho(n) \in V_{\lambda+k\alpha} = V_{\sigma(\lambda)}$ for some integer k, which means $\sigma(\lambda) = \lambda + k\alpha$ for some $k \in \mathbb{Z}$, as required. □

Recall from Theorem 3.5.1.5 that Φ is an abstract root system in the vector space $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}(T)$. Note also that a consequence of this is that elements of W can be considered as transformations of this vector space.

Definition 4.1.2.3. An element $\beta \in X(T)$ is called an *abstract weight* if, when considered as a vector in $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}(T)$, it has the property that, for all $\alpha \in \Phi$, $\sigma_\alpha(\beta) - \beta$ is an integer multiple of α.

Proposition 4.1.2.4. *The weights of a rational representation are abstract weights.*

Proof. This follows immediately from Proposition 4.1.2.2. □

4.1.3 The Big Cell

Proposition 4.1.3.1. *Let $B^- = TU^-$ be opposite to B. The product map*

$$\pi : U^- \times B \longrightarrow G$$

defines a bijection of $U^- \times B$ onto a dense open subset $\Omega \subset G$.

Proof. If $u, u' \in U^{-1}$ and $b, b' \in B$ are such that $\pi(u, b) = ub = u'b' = \pi(u', b')$, implies $u^{-1}u' = bb'^{-1}$ but $U^{-1} \cap B = \{e\}$, so $u = u', b = b'$, which implies π is injective.

Now, since π is a morphism of varieties, $\mathrm{im}\pi = \Omega$ is constructible, and therefore its closure contains a dense open set V. Since G is connected, we can apply Proposition 1.3.1.6, which says V is dense in G, and so

$$G = \overline{V} = \overline{\Omega}$$

and so Ω is dense in G. Suppose now that $\sigma \in W$ sends B^- to B. That is, $B^- = n^{-1}Bn$. Then

$$\Omega = U^-B = U^-TB = B^-B = n^{-1}BnB$$

so $n\Omega = B\sigma B$, which means Ω is open in G if and only $B\sigma B$ is. Let $q : G \to G/B$ be the quotient morphism, and let $x_0 \in G/B$ correspond to B. Since morphisms of varieties are continuous, $B\sigma B = q^{-1}(Bn.x_0)$ is open if $Bn.x_0$ is. But $Bn.x_0$

is the B-orbit of $n.x_0$, and so by Proposition 1.4.3.3 $Bn.x_0$ is locally closed. Applying Lemma 1.4.3.2 gives us that $Bn.x_0$ is open in its closure.

Now, since q is surjective, $q^{-1}(\overline{Bn.x_0})$ is closed in G. But

$$B\sigma B = q^{-1}(Bn.x_0) \subset q^{-1}(\overline{Bn.x_0})$$

and so $\overline{B\sigma B} \subset q^{-1}(\overline{Bn.x_0})$. But we saw above that $\overline{B\sigma B} = G$, and since q is surjective, we conclude from the above equation that

$$G/B = q(\overline{B\sigma B}) \subset qq^{-1}(\overline{bn.x_0}) = \overline{Bn.x_0}$$

which means that $Bn.x_0$ is in fact open in G/B, and so Ω is open in G, as required.

□

Definition 4.1.3.2. Call $\Omega \subset G$ in Proposition 4.1.3.1 the *big cell* of G.

The following proposition (which is in fact a consequence of Zariski's Main Theorem, a very significant theorem on birational morphisms of varieties and their fibres), is given without proof. For the details, see §28.5 of [6].

Proposition 4.1.3.3. *The map above*

$$\pi : U^- \times B \longrightarrow \Omega$$

is in fact an isomorphism of varieties.

Example 38. Let $G = SL(2,K), T = D(2,K), B = T(2,K)$. Then $n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N_G(T)$ sends B to the other Borel subgroup containing T, namely the group of invertible lower triangular matrices of determinant one. Therefore

$$U^- = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

for $x \in K$. Since $\Omega = U^-B$, elements of Ω are of the form

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ ax & bx + a^{-1} \end{pmatrix}$$

so $\Omega \subset \{g \in G \,|\, g_{11} \neq 0\}$. On the other hand, if $x \in G$ such that $g_{11} \neq 0$,

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ g_{21}g_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ x & g_{11}^{-1} \end{pmatrix}$$

since $g_{22} = (g_{12}g_{21} + 1)g_{11}^{-1}$. Therefore, $\Omega = \{g \in G \,|\, g_{11} \neq 0\}$. Indeed, let $f = T_{11} \in K[G]$. Then $\Omega = G_f$, so Ω is in fact a principal open subset of G.

4.2 Modules

4.2.1 Maximal Vectors

From this point on, we will assume that the G-modules V under discussion are not equal to 0.

Definition 4.2.1.1. Let $\rho : G \to GL(V)$ be a representation. A nonzero element $v \in V$ is called a *maximal vector* of ρ if the subspace Kv is stable under $\rho(B)$.

Note that this definition is dependent on the choice of B.

Lemma 4.2.1.2. *If $\rho : G \to GL(V)$ is a representation, then ρ has a maximal vector $v \in V$.*

Proof. Certainly $\rho(B)$ is a connected and solvable subgroup of $GL(V)$, so the Lie-Kolchin theorem says $\rho(B)$ has a common eigenvector $v \in V$. That is, Kv is stable under $\rho(B)$ and so v is a maximal vector of ρ. □

Lemma 4.2.1.3. *Given a representation $\rho : G \to GL(V)$, a nonzero element $v \in V$ is a maximal vector of ρ if and only if $v \in V_\lambda$ for some $\lambda \in \Lambda$, and $\rho(U_\alpha)$ fixes v for all $\alpha \in \Phi^+$.*

Proof. Suppose firstly that $v \in V$ is a maximal vector of ρ. Since $T \subset B$, it follows that $\rho(t)v = \lambda(t)v$ for each $t \in T$, where $\lambda : T \to \mathbb{G}_m$ is the coordinate function. Since this function is linear, it is immediate that λ is a G-morphism, and therefore lies in $\mathrm{X}(T)$ and, moreover, $v \in V_\lambda$, whence $\lambda \in \Lambda$. Suppose now that $u \in U_\alpha$ for some $\alpha \in \Phi^+$. Since $U_\alpha \subset B$, we have $\rho(u)v = kv$ for some $k \in K$. That is, v is a k-eigenvector for $\rho(u)$. But u is unipotent, and its only eigenvalue is 1. Therefore $\rho(u)v = v$ as required.

On the other hand, suppose these conditions hold for v. We wish to show that v is a maximal vector. Since G is semisimple, so is B, and so by Theorem 3.4.5.1(4), B is generated by T and the groups U_α for $\alpha \in \Phi^+$. But by assumption, $\rho(U_\alpha)v = v$ for all positive α, and $\rho(T)v \subset Kv$, which means $\rho(B)$ stabilises Kv, as required. □

Definition 4.2.1.4. Let $\rho : G \to GL(V)$ be a representation with maximal vector $v \in V$. By Lemma 4.2.1.3, $v \in V_\lambda$ for some $\lambda \in \Lambda$. We call λ the *weight* of v.

Proposition 4.2.1.5. *Let $\rho : G \to GL(V)$ be a nontrivial rational representation, with maximal vector v^+ of weight λ. Let V' be the G-submodule of V spanned by the orbit space $G.v^+$. Then λ has multiplicity 1 in V', and the other weights of V' are of the form*

$$\lambda - \sum c_\alpha \alpha$$

where $\alpha \in \Phi^+, c_\alpha \in \mathbb{Z}^+$. Moreover, V' has a unique maximal submodule.

Proof. We saw in the proof of Corollary 3.5.1.4 that for any $\alpha \in \Phi$,

$$\rho(U_\alpha)V_\lambda \subset \sum V_{\lambda+k\alpha}$$

for $k \in \mathbb{Z}^+$. In particular, if $\alpha \in \Phi^+$, $x \in U_{-\alpha}$ then

$$\rho(x).v^+ = v^+v'$$

where $v' \in \sum V_{\lambda-k\alpha}$, for k a positive integer, while if $x \in U_\alpha$ for $\alpha \in \Phi^+$, then since v^+ is a maximal vector, $\rho(x).v^+ = v^+$.

Since U^- is generated by $U_{-\alpha}$ for $\alpha \in \Phi^+$, it follows that elements of $U^-.v^+$ are sums of vectors of weight

$$\lambda - \sum_{\alpha\in\Phi^+} k_\alpha \alpha$$

where $k_\alpha \in \mathbb{Z}^+$, and moreover if $x \in U^-$ is such that $\rho(x).v^+ \in V_\lambda$, then $\rho(x).v^+ = v^+$.

Since v^+ is a maximal vector,

$$B.v^+ \subset Kv^+$$

This fact, combined with Proposition 4.1.3.3, gives

$$\operatorname{span}(U^-.v^+) = \operatorname{span}(U^-B.v^+) = \operatorname{span}(\Omega.v^+) \tag{4.1}$$

By Proposition 4.1.3.1, Ω is dense in G, and so, since linear spaces are closed,

$$\operatorname{span}(\Omega.v^+) = \operatorname{span}(G.v^+) = V' \tag{4.2}$$

Equations (4.1) and (4.2) together tell us that

$$V' = V_\lambda \oplus \bigoplus V_{\mu_i} \tag{4.3}$$

where each μ_i is of the form $\lambda - \sum_{\alpha\in\Phi^+} k_\alpha\alpha$ for $k_\alpha \in \mathbb{Z}^+$. Moreover, we have shown that the only elements in V' of weight λ are scalar multiples of v^+, which means that λ is of multiplicity 1 in V', that is, $\dim V_\lambda = 1$.

We now need to identify a maximal proper submodule of V'. In the first place, let $V'' \subset V'$ be a submodule. Note that, if V'' contains a nonzero vector of weight λ then $v^+ \in V''$, whence

$$V' = \operatorname{span}(G.v^+) \subset \operatorname{span}(G.V'') \subset V''$$

On the other hand, if V'' does not contain a nonzero vector of weight λ, then in particular $v^+ \notin V''$ and so obviously $V'' \neq V'$. What we have shown then is that, given a submodule $V'' \subset V'$, this inclusion is proper submodule if and only if V'' contains no vectors of weight λ. Note that if $V' = V_\lambda$, then the only proper submodule is $\{0\}$, which is certainly maximal. The sum of all proper submodules therefore shares this property of containing no vectors of weight λ, and therefore is the unique maximal submodule of V'.

□

Definition 4.2.1.6. Call λ in Proposition 4.2.1.5 the *highest weight* of V'.

The motivation for this definition is the natural partial order on weights, dependent on Δ:

$$\lambda \geq \mu \qquad \text{if } \lambda - \mu = \sum_{\alpha \in \Phi^+} c_\alpha \alpha, \text{ for } c_\alpha \in \mathbb{Z}^+$$

The proposition shows, then, that $\lambda \geq \mu$ for all weights μ of V'.

4.2.2 Dominant Weights

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be the base for Φ corresponding to the Borel subgroup B.

Definition 4.2.2.1. Define an abstract weight λ_i, for $1 \leq i \leq l$, by setting

$$\lambda_i - \sigma_{\alpha_j}(\lambda_i) = \delta_{ij}\alpha_j$$

and call it a *fundamental dominant weight.* Call an abstract weight λ a *dominant weight* if $\lambda = \sum c_i \lambda_i$ for $c_i \in \mathbb{Z}^+$.

It is a fact which we will not prove here that the set of abstract weights is a free abelian group of rank l, and that the set of fundamental dominant weights $\{\lambda_1, \ldots, \lambda_l\}$ is a basis for this group.

Proposition 4.2.2.2. *If λ is the highest weight of a G-module V', then λ is a dominant weight.*

Proof. It is a fact that each weight of a module is W-conjugate to exactly one dominant weight, which means there is an element $\sigma \in W$ such that $\sigma(\lambda)$ is dominant. It furthermore the case fact that, if μ is dominant, then $\mu \geq \tau(\mu)$ for all $\tau \in W$, which means

$$\sigma(\lambda) \geq \sigma^{-1}(\sigma(\lambda)) = \lambda$$

But λ is the highest weight, which means $\lambda = \sigma(\lambda)$, and so it is dominant. $\square$

4.2.3 Highest Weight Modules

Theorem 4.2.3.1. *Let V be an irreducible G-module. Then*

1. *There is a unique B-stable 1-dimensional subspace, spanned by a maximal vector of some dominant weight λ, whose multiplicity is 1. λ is the highest weight of V.*

2. *All other weights of V are of the form $\lambda - \sum k_\alpha \alpha$ where $\alpha \in \Phi^+$ and $c_\alpha \in \mathbb{Z}^+$. They are permuted by W, with W-conjugate weights having the same multiplicity.*

3. *If V' is another irreducible G-module of highest weight λ', then V is isomorphic to V' as a G-module, if and only if $\lambda = \lambda'$.*

Proof. We can apply Lemma 4.2.1.2 to find a maximal vector $v^+ \in V$ of weight λ, and by Proposition 4.2.1.5 the span of the orbit space $G.v^+$, call it V', has weights $\lambda - \sum_{\alpha\in\Phi^+} k_\alpha \alpha$, and such that $V' \cap \dim V_\lambda = 1$. But since V is irreducible and V' is nonzero, it follows that $V = V'$, and V_λ is a suitable choice for the B-stable subspace in (1).

Suppose that there is another maximal vector, $w^+ \in V$, of weight μ. Applying Proposition 4.2.1.5 again tells us that $G.w^+ = V$, and, moreover, that μ is the highest weight of V. In particular, $\mu \geq \lambda$. But we have already determined above that λ is also a highest weight vector of V, and so $\lambda \geq \mu$, from which we must conclude that $\lambda = \mu$. It is now immediate that $V_\lambda = V_\mu$, the uniqueness property which completes the proof of (1).

We have already seen that the weights of V are of the form $\lambda - \sum_{\alpha\in\Phi^+} k_\alpha \alpha$ for $k_\alpha \in \mathbb{Z}^+$, and Proposition 4.1.1.2 tells us that W permutes the weights of V, and $\dim V_\mu = \dim V_{\sigma(\mu)}$ for all weights μ and elements $\sigma \in W$, which completes the proof of (2).

We now turn to (3). Suppose firstly that $f : V \to V'$ is an isomorphism. Then, for $t \in T$,

$$t.f(v^+) = f(t.v^+) = f(\lambda(t)v^+) = \lambda(t)f(v^+)$$

and so λ is a weight of V', therefore $V_\lambda \subset V'$ is B-stable and 1-dimensional, with λ dominant. But (1) tells us that such submodules are unique, which means $V_\lambda = V_{\lambda'}$, and so $\lambda = \lambda'$.

On the other hand, suppose $\lambda = \lambda'$. Consider the G-module $Y = V \oplus V'$. Let $v \in V, v' \in V'$ be maximal vectors, then $v + v'$ is a maximal vector of Y. Indeed, if $t \in T$, then

$$t.(v + v') = t.v + t.v' = \lambda(t)v + \lambda'(t)v = \lambda(v + v')$$

since $\lambda = \lambda'$. It follows that $v + v'$ is a maximal vector of weight λ, and so, by Proposition 4.2.1.5, $G.(v + v') = Y'$ is a G-module whose weights are of the form $\lambda - \sum k_\alpha \alpha$, and where $\dim(Y' \cap Y_\lambda) = 1$. Of course, $v, v' \in Y$ are both themselves of weight λ. Since v and v' are nonproportional by construction, they are also nonproportional to $v + v'$, and therefore cannot lie in $Y' \cap Y_\lambda$, since it is of dimension 1. Therefore, $v, v' \notin Y'$. In particular, this implies $V \cap Y' \subsetneq V$ and $V' \cap Y' \subsetneq V'$, but V, V' are assumed to be irreducible, which implies $V \cap Y' = V' \cap Y' = \{0\}$. Consider now the projection maps

$$\begin{aligned} p_1 &: Y' \longrightarrow V \\ p_2 &: Y' \longrightarrow V' \end{aligned}$$

Since $V \cap Y' = V' \cap Y' = \{0\}$, we conclude p_1, p_2 are injective. Moreover, $p_1(v+v') = v, p_2(v+v') = v'$, and so the images of p_1, p_2 are nonzero submodules of V, V' respectively. By irreducibility again, we have $p_1(Y') = V, p_2(Y') = V'$, and so p_1, p_2 are surjective. We have shown, then, that $Y' = V = V'$, as required. □

Theorem 4.2.3.1 tells us that an irreducible G-module can be parametrised by its (unique) highest weight λ.

4.2.4 Construction of Highest Weight Modules

Lemma 4.2.4.1. *If there is a G-module V which contains a maximal vector v of weight λ, then there is an irreducible G-module W of highest weight λ.*

Proof. Let V' be the G-module spanned by $G.v$. Proposition 4.2.1.5 tells us that there is a maximal submodule $V'' \subset V'$ with the property $V_\lambda \cap V'' = \{0\}$. Since V'' is maximal, it is immediate that the quotient module V'/V'' is an irreducible G-module which contains v. Passing to the quotient does not alter the fact that v, when considered as an element of V'/V'', is a maximal vector of weight λ, and so V'/V'' serves as the required G-module W. $\square$

Theorem 4.2.4.2. *Let $\lambda \in \mathrm{X}(T)$ be a dominant weight of $\rho : G \to GL(V)$. Then there exists an irreducible G-module of highest weight λ.*

The proof of Theorem 4.2.4.2 requires the following two lemmas:

Lemma 4.2.4.3. *Suppose λ is the highest weight of an irreducible G-module V and λ' is the highest weight of an irreducible G-module V'. Then $\lambda + \lambda'$ is the highest weight of an irreducible G-module.*

Proof. By Proposition 4.2.1.5, we can find maximal vectors $v \in V$ and $v' \in V$ of weight λ and λ' respectively. Consider the tensor product $V \otimes V'$, and let $t \in T$. Then

$$t(v \otimes v') = t.v \otimes t.v' = \lambda(t)v \otimes \lambda'(t)v' = \lambda(t)\lambda'(t)(v \otimes v') = (\lambda + \lambda)(t)(v \otimes v')$$

and so $v \otimes v'$ is a vector of weight $\lambda + \lambda'$. Suppose now that $x \in U_\alpha$. Then

$$x(v \otimes v') = x.v \otimes t.v' = v \otimes v'$$

By Lemma 4.2.1.3, these two facts show that $v \otimes v' \in V \otimes V'$ is a maximal vector of weight $\lambda + \lambda'$. By Lemma 4.2.4.1, then, there exists an irreducible G-modules W of highest weight $\lambda + \lambda'$. $\square$

Lemma 4.2.4.4. *Let $\lambda \in \mathrm{X}(T)$ be a dominant weight. Then there exists a positive integer d such that $d\lambda$ is the highest weight of an irreducible G-module.*

Proof. For each $i \in \{1, \ldots, l\}$, consider the subgroup

$$P_i = BW_{\Delta - \{\alpha_i\}}B$$

where $W_{\Delta - \{\alpha_i\}}$ is the subgroup of W generated by the reflections $\{\sigma_{\alpha_k} \mid k \neq i\}$. We saw in the proof of Proposition 4.1.3.1 that $B\sigma_{\alpha_i}B$ is open, and so, by Theorem 3.6.3.1, P_i is closed. It follows also that P_i is parabolic, since it contains B. Using Theorem 1.5.1.2, there exists a rational representation $\rho_i : G \to GL(V_i)$ and a line $Kv_i \subset V_i$ such that

$$P_i = \{g \in G \mid \rho_i(g)(Kv_i) = Kv_i\} \tag{4.4}$$

In particular, $\rho_i(B)v_i \subset Kv_i$, and so v_i is a maximal vector of ρ_i.

By Proposition 4.2.1.5 combined with Lemma 4.2.4.1, there is an irreducible G-submodule V_i' of V_i, containing v_i as a weight vector of highest weight μ_i, say. Since μ_i is dominant, we can express it in the form $\mu_i = \sum d_j\lambda_j$ for some $d_j \in \mathbb{Z}^+$, where the λ_j's are the fundamental dominant weights.

Now, by construction, for each for $k \neq i$, the reflection σ_{α_k} has a representative $n_k \in P_i$. By the equality in (4.4), $\rho_i(n_k)v_i \in Kv_i$, and in particular, $\rho_i(n_k)v_i$ is of weight μ_i. But by Proposition 4.1.1.2, σ_{α_k} sends V_{μ_i} to $n_kV_{\mu_i}$, which we just saw is simply V_{μ_i}. That is, $\sigma_{\alpha_k}(\mu_i) = \mu_i$ for all $k \neq i$.

Putting this all together, for $k \neq i$ we get

$$\begin{aligned}
\mu_i = \sigma_{\alpha_k}(\mu_i) &= \sigma_{\alpha_k}\big(\sum_{j=1}^{l} d_j\lambda_j\big) \\
&= \sum_{j=1}^{l} d_j\sigma_{\alpha_k}(\lambda_k) \\
&= d_k\,(\lambda_k - \alpha_k) + \sum_{j\neq k} d_j\lambda_j \\
&= -d_k\alpha_k + \sum_{j=1}^{l} d_j\lambda_j \\
&= \mu_i - d_k\alpha_k
\end{aligned}$$

and so $d_k = 0$ for all $k \neq i$. But μ_i is nonzero, so $\mu_i = d_i\lambda_i$ for some nonzero $d_i \in \mathbb{Z}^+$.

Define $d = \prod d_i$. Since λ is assumed to be dominant, we have $\lambda = \sum c_j\lambda_j$ for some $c_i \in \mathbb{Z}^+$. Then

$$d\lambda = d\big(\sum_i c_i\lambda_i\big) = \sum_i c_id\lambda_i = \sum_i c_i\big(\prod_{j\neq i} d_j\big)d_i\lambda_i$$

Setting $k_i = c_i\big(\prod_{j\neq i} d_j\big)$, this gives us

$$d\lambda = \sum_i k_i\mu_i$$

where $k_i \in \mathbb{Z}^+$. Now we have already found irreducible G-modules V_i' of highest weight μ_i, and so repeated applications of Lemma 4.2.4.3 tells us that there exists an irreducible G-module of highest weight $d\lambda$, as required. □

Proof of Theorem 4.2.4.2. Suppose firstly that we already have an irreducible G-module V of highest weight λ. We wish to find an irreducible submodule which is a quoteient of $K[G]$ (considered as a G-module by the action of right translation) and which is of highest weight λ, and therefore is isomorphic to V. Let $v^+ \in V_\lambda$ be a maximal vector, so $V_\lambda = Kv^+$, and consider the above vector space decomposition

$$V = V_\lambda \oplus V' \tag{4.5}$$

where V' is the space spanned by weight vectors of weight other than λ. We define a linear functional $f \in V^*$ as follows: $f(v^+) = 1$, while if $v' \in V'$, then $f(v) = 0$. We now define an element $c_\lambda \in K[G]$: for $x \in G$,

$$c_\lambda(x) = f(x.v^+)$$

Since $x.v^+ \in V$, we can use (4.5) to express it as $x.v^+ = av^+ + v'$ for some $a \in K$ and $v' \in V'$. Note now that

$$c_\lambda(x) = f(x.v^+) = f(av^+ + v') = af(v^+) + f(v') = a$$

and so $c_\lambda(x)$ gives the v^+ coordinate of $x.v^+$, as it were. More formally,

$$c_\lambda(x)v^+ = av^+ = x.v^+ - v'$$

from which we conclude $x.v^+ \equiv c_\lambda(x)v^+ \pmod{V'}$. Now $\lambda \in \mathrm{X}(T)$, but we can extend it to a homomorphism on B by defining $\lambda(U) = 1$. That is, for $z \in B$, we have $z = tu$ for some $t \in T, u \in U$, and then $\lambda(z) = \lambda(tu) = \lambda(t)$. In fact, note that $z.v^+ = tu.v^+ = t.v^+ = \lambda(t)v^+$. Similarly, we can extend it to a homomorphism on B^- by defining $\lambda(U^-) = 1$, so for $x = t'u' \in B^-$, with $t' \in T, u' \in U^-$, we have $\lambda(x) = \lambda(t'u') = \lambda(t')$. Note that $x.v^+ = t'u'.v^+ = t'.(v^+ + v') = \lambda(t') + v''$ where $v', v'' \in V'$.

Suppose now that $x = t'u' \in B^-$ and $z \in tu \in B$ as above, and also that $y \in G$. Then, by using the above observations gives

$$xyz.v^+ = \lambda(z)xy.v^+ = c_\lambda(y)\lambda(z)x.(v^+ + v') = \lambda(x)c_\lambda(y)\lambda(z)v^+ + v''$$

where $v', v'' \in V'$. Therefore

$$c_\lambda(xyz) = f(xyz.v^+) = f\big(\lambda(x)c_\lambda(y)\lambda(z)v^+ + v''\big) = \lambda(x)c_\lambda(y)\lambda(z)$$

In particular, if $x \in U^-$ and $y \in B$, then this formula gives us $c_\lambda(xy) = \lambda(y)$. By Proposition pbigcell, we see that this formula determines c_λ on the dense open subset $\Omega \subset G$, and so c_λ is completely determined by λ. Moreover, if $y \in B$ and $z \in G$, then under the action of G on $K[G]$ by right translation,

$$(y.c_\lambda)(z) = (\rho_y c_\lambda)(z) = c_\lambda(zy) = c_\lambda(z)\lambda(y)$$

which is to say $y.c_\lambda = \lambda(y)c_\lambda$, and so c_λ is stable under B and is a vector of weight λ. This means that c_λ is a maximal vector of $K[G]$. By Lemma 4.2.4.1, there exists an irreducible G-module of highest weight λ which is a quotient of $K[G]$, which is what we wanted.

We no longer need suppose that an irreducible G-module V of weight λ exists. Instead, we simply define a polynomial function $c \in K[G]$ as follows. In the first place, if $x \in U^-, y \in B$, then set $c(xy) = \lambda(y)$, where λ is extended to a homomorphism on B in the same fashion as above. Since $\Omega \subset G$ is open by Proposition 4.1.3.1, $K(\Omega) = K(G)$, and so c is contained in $K(G)$, that is, it is a rational function on G. It is now sufficient to show that some positive power of c is everywhere defined on G, since $K[G] = \bigcup_{z\in G} .\mathcal{O}_z$. For if we have such a c^k, suppose for a contradiction that c is not defined at z. Then Theorem 5.3B of [6] says we can find a rational function f' on a subvariety Y of G which contains z, such that $f' = 1/c \in \mathcal{O}_{z'}$ for some $z' \in Y$ and such that f' is equal to zero wherever it is defined on Y. In particular, $f'(z) = 0$. But then $f^k(z) = 0 = 1/c^k(z)$, which implies c^k is not defined at z, which is contrary to our assumption that c^k is everywhere defined. Therefore it follows that c is everywhere defined.

By Lemma 4.2.4.4, there is a positive integer d such that $d\lambda$ is the highest weight of an irreducible G-module V. Then using the argument advanced in the beginning of this proof, V is isomorphic to a G-module V' with maximal vector $c_{d\lambda} \in K[G]$ of weight $d\lambda$. Moreover, for $x \in U^-, y \in B$,

$$c_{d\lambda}(xy) = d\lambda(y) = \lambda^d(y) = c^d(xy)$$

which means $c_{d\lambda} = c^d$ on Ω. We therefore extend c^d to G by setting $c^d(z) = c_{d\lambda}(z)$ for all $z \in G$. Then c is defined on all of G, and so lies in $K[G]$. As above, if we consider $K[G]$ as a G-module under the action of right translation, then for $y \in B$ and $z \in G$,

$$y.c(z) = (\rho_y c)(z) = c(zy) = \lambda(y)c(z)$$

which means c is a maximal vector of weight λ, and so another application of Lemma 4.2.4.1 tells us that there is an irreducible G-module of highest weight λ, as required. □

Example 39. Let $G = SL(2,K)$. We saw in Example 24 that G is semisimple. Let $T = D(2,K)$ and $B = T(2,K)$. Then $\alpha : \operatorname{diag}(t,t^{-1}) \to t/(t^{-1}) = t^2$ is the only positive root, and the corresponding fundamental dominant weight is $\lambda = \frac{1}{2}\alpha$, that is

$$\lambda : \operatorname{diag}(t,t^{-1}) \to t$$

Any character $n\lambda = \frac{n\alpha}{2}$ with $n \in \mathbb{Z}^+$ is therefore a dominant weight. The theorem says that there is an irreducible G-module which has $n\lambda$ as its highest weight.

Indeed, set $V = \mathscr{P}^n(K^2)$ be the vector space of homogeneous polynomials of degree n over K^2. Then V is of dimension $n+1$. Indeed, if we define vectors $v_k \in V$,

$$v_k(T_1,T_2) = T_1^{n-k}T_2^k$$

then $\{v_0,\dots,v_n\}$ is a basis for V. Now define a representation $\rho : G \to GL(V)$ as follows:

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(T_1,T_2) = f\left((T_1,T_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f(aT_1 + cT_2, bT_1 + dT_2)$$

In particular, for $\operatorname{diag}(t,t^{-1}) \in T$,

$$\rho(\operatorname{diag}(t,t^{-1}))f(T_1,T_2) = f(tT_1, t^{-1}T_2)$$

so

$$\rho(\mathrm{diag}(t,t^{-1}))v_0 = t^n v_0 = (n\lambda)(t)v_0$$

which means v_0 is a vector of weight $n\lambda$. Moreover, for $x = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in U_\alpha$,

$$\rho(x)f(T_1,T_2) = f(T_1, bT_1 + T_2)$$

and in particular,

$$\rho(x)v_0 = T_1^n(bT_1 + T_2)^0 = T_1^n = v_0$$

so v_0 is a maximal vector of weight $n\lambda$. We can also make the remark that

$$\rho(\mathrm{diag}(t,t^{-1}))v_k = (tT_1)^{n-k}(t^{-1}T_2)^k = t^{n-2k}v_k = \big((n-2k)\lambda\big)(t)v_k$$

and so $v_k \in V_{(n-2k)\lambda}$.

Chapter 5

Classical Algebraic Groups

This chapter is in a sense the culmination of the paper, and what we have been working towards. The Classical Groups serve as the canonical examples of semisimple groups, and are therefore the perfect testing ground to apply the extensive theory which we have gathered in the preceeding chapters.

We work through these examples in the greatest detail, ensuring that the full extent of the structure is revealed in each case. In particular, we give representative maximal tori, Borel subgroups and give the root space decomposition and Weyl group in each case. In addition, the Dynkin diagram of each root system is calculated, as a further way of classifying each of the Classical Groups.

Indeed, we will discover that the Classical Groups exhaust all of the infinite classes of root systems, namely the classes A_l, B_l, C_l and D_l. This of course means that the Classical Groups account for 'most' of the reductive groups we can think of – any others must either correspond to the finite class of exceptional root systems, or be built up from the Classical Groups. The parameter l in the following describes the rank of the group in question.

5.1 The General Linear Group

First we consider the group of invertible $l \times l$ matrices $G = GL(l, K)$.

5.1.1 Maximal Torus

Let T be the group $D(l, K)$, that is the group of $l \times l$ diagonal matrices. Then by definition T is a torus, and we wish to show it is in fact a maximal torus. Suppose, for a contradiction, that $T' \subset GL(l, K)$ is torus with $T \subsetneq T'$. In particular, this means $\dim T < \dim T'$. Now, by Proposition 2.1.1.2, there exists an element $x \in GL(l, K)$ such that $x^{-1}T'x \subset T$. But certainly $T' \cong x^{-1}T'x$, and so

$$\dim T' = \dim(x^{-1}T'x) \leq \dim T$$

which is a contradiction, and so we conclude that T is maximal.

5.1.2 Borel Subgroup

Let $B = T(l, K)$. We know that B is closed, connected and solvable. We wish to show that it is maximal under these conditions, so suppose, for a contradiction, that $B' \subset GL(l, K)$ is closed, connected and solvable with $B \subsetneq B'$. In particular, this means $\dim B < \dim B'$. Now, by the Lie-Kolchin Theorem, there exists an element $x \in GL(l, K)$ such that $x^{-1}B'x \subset B$. But certainly $B' \cong x^{-1}B'x$, and so

$$\dim B' = \dim(x^{-1}B'x) \leq \dim B$$

which is a contradiction, and so we conclude that B is maximal.

5.1.3 Radical and Unipotent Radical

As above, set $B = T(l, K) \subset GL(l, K)$. Consider now the group of lower triangular matrices, which we will call L. It is easy to see that $JLJ^{-1} = B$, where

$$J = \begin{pmatrix} 0 & & 0 & 1 \\ 0 & & 1 & 0 \\ & \iddots & & \\ 1 & & 0 & 0 \end{pmatrix} \in GL(l, K)$$

Note that J is a permutation matrix which corresponds to the 'order reversing' element of the symmetric group $\mathfrak{S}_n$:

$$(1, n)(2, n-1) \cdots$$

Now, L is evidently also a Borel subgroup which contains T. In the notation of §3.4.1, where $I(T)$ denotes the connected component of the intersection of all Borel subgroups containing T, we have shown the inclusion $I(T) \subset B \cap L$. Indeed,

$$T \subset I(T) \subset B \cap L = T$$

from which we get $I(T) = T$. But Corollary 3.4.1.4 says $I(T) = T \cdot R_u(G)$, and so we must conclude that $R_u(G) \subset T$, which forces $R_u(G) = \{e\}$. Therefore $GL(l, K)$ is reductive, but not semisimple.

Moreover, by Lemma 3.1.1.9, $R(G) = [Z(G)]^\circ$. It is easy to show (using Schur's Lemma again, for example) that the centre of G consists of scalar matrices. That is, $Z(G) \cong \mathbb{G}_m$, and this group is connected, from which we conclude that $R(G) \cong \mathbb{G}_m$.

5.1.4 Root System

It is a standard calculation to show that, for $G = GL(l, K)$, we have that $\mathfrak{g} = \mathscr{L}(G) = \mathrm{Mat}(l, K)$. See, for example, §1.2 of [4]. Moreover, $\mathscr{L}(T) = \mathfrak{d}(l, K)$ is the Lie algebra of diagonal matrices.

Now, let $\mathsf{x} \in \mathrm{Mat}(l+1, K)$. Then, for $t = \mathrm{diag}(t_1, \ldots, t_l) \in T$, we have

$$\left(t\mathsf{x}t^{-1}\right)_{rs} = (t_r/t_s)\mathsf{x}_{rs}$$

for $1 \leq r, s \leq l$. It follows that, if $\mathsf{x} = E_{i,j}$, that is, the matrix with 1 in the $(i,j)^{\text{th}}$ entry and 0's elsewhere, we have

$$\mathsf{t}\mathsf{x}\mathsf{t}^{-1} = (t_i/t_j)\mathsf{x} \tag{5.1}$$

for arbitrary $t \in T$. This tells us that, for $1 \leq i, j \leq l$, with $i \neq j$, we have $E_{i,j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l) \longmapsto t_i/t_j \tag{5.2}$$

Since $GL(l, K)$ is reductive, $\mathfrak{g}_\alpha$ is of dimension 1, and so

$$\mathfrak{g}_\alpha = \{cE_{i,j} \mid c \in K\}$$

Since G is reductive, we can use Theorem 3.4.3.2(5) to equate the Lie algebras

$$\mathfrak{g} = \mathfrak{t} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

By comparison of dimensions, the $l(l-1)$ roots of the form given in equation (5.2) comprise all the roots of $GL(l, K)$.

It therefore follows that the roots of $GL(l, K)$ are the characters

$$\operatorname{diag}(t_1, \ldots, t_l) \longmapsto t_i t_j^{-1}, \quad 1 \leq i, j \leq l,\ i \neq j \tag{5.3}$$

5.1.5 Weyl Group

To calculate $W(G, T)$, with $T = D(l, K)$, we first calculate the normaliser $N_G(T)$.

Let p_σ be the $l \times l$ permutation matrix corresponding to $\sigma \in \mathfrak{S}_l$, the symmetric group on l elements. That is, $(p_\sigma)_{ij} = \delta_{\sigma(i),j}$. Then, for any element $s \in T$, the matrix $x = p_\sigma s$ is monomial. Moreover, given an element $t \in T$, we have

$$xtx^{-1} = p_\sigma st(p_\sigma s)^{-1} = p_\sigma sts^{-1}p_\sigma^{-1}$$

which, it is easy to see, is diagonal. Therefore, monomial matrices normalise T.

On the other hand, suppose $x \in N_G(T)$. Then, for all $t \in T$, we have $xtx^{-1} = s$ for some $s \in T$. That is, $x = s^{-1}xt$, and so

$$x_{ij} = (t_j/s_i)x_{ij}$$

Suppose $x_{ij} \neq 0$. Then clearly, for the above equality to hold, we must have $s_i = t_j$. Suppose now k is such that $x_{ik} \neq 0$. Then, likewise, $s_i = t_k$. But then $t_i = t_k$, and this must hold for arbitrary $t \in T$, from which we must conclude $j = k$, which is to say that each row of x contains at most 1 nonzero element. But since x is invertible, each row of x must contain at least 1 nonzero element. The conclusion is that each row of x has precisely 1 nonzero element, that is, x is monomial. We have shown, then, that $N_G(T)$ consists of the monomial matrices.

Certainly it is true that each monomial matrix x is of the form $p_\sigma t$ for some $t \in T$, and so each element $xT \in N_G(T)/T = W$ can be rewritten as $xT = p_\sigma T$. It follows immediately that $W \cong \mathfrak{S}_l$, via the correspondence $p_\sigma \leftrightarrow \sigma$.

As a final remark on the group $GL(l, K)$, we will observe below that the Weyl group and the set of roots for $GL(l, K)$ coincide with those of $SL(l+1, K)$.

5.2 The Special Linear Group

As we have seen, the group $SL(l+1, K)$ is defined to be that which consists of matrices of determinant equal to 1 in $GL(l+1, K)$. This is certainly a group, and it is closed, since

$$SL(l+1, K) = \{x \in GL(l+1, K) \mid \det(x) - 1 = 0\}$$

We also saw in Proposition 1.3.2.8 that G is connected.

5.2.1 Radical and Unipotent Radical

Write $G = SL(l+1, K)$. If $g \in G$ and $x \in GL(l+1, K)$, then

$$\det(g) = \det(xyx^{-1}) = \det(x)\det(y)\det(x^{-1}) = 1$$

and so $g \in G$. Therefore G is a closed, normal subgroup of $GL(l+1, K)$. By Corollary 3.1.1.11, it follows that G is reductive, and so $R_u(G) = \{e\}$.

Moreover, by Lemma 3.1.1.9, $R(G) = Z(G)^\circ$, but the centre of G consists of scalar matrices of determinant one. In particular, the entries of such a matrix must consist of the finitely many $(l+1)^{\text{th}}$ roots of unity, and so $Z(G)$ is finite, and therefore its connected component is trivial. We have shown, then, that $SL(l+1, K)$ is semisimple.

5.2.2 Maximal Torus

Retain the notation $G = SL(l+1, K)$, and let $T = G \cap D(l+1, K)$. Then T consists of elements of the form

$$t = \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_l & \\ & & & (t_1 \cdots t_l)^{-1} \end{pmatrix} \tag{5.4}$$

It is clear that T is a torus, since it is isomorphic to $D(l, K)$. We now wish to calculate the centraliser of T, so suppose $x \in G$ such that, for all $t = \operatorname{diag}(t_1, \ldots, t_{l+1}) \in T$, we have $txt^{-1} = x$. In particular, in terms of the $(i, j)^{\text{th}}$ entry

$$(txt)_{ij} = (t_i/t_j)x_{ij} = x_{ij}$$

But this equation must hold for all t, and so it follows $x_{ij} = 0$ if $i \neq j$, that is, x is diagonal. On the other hand, since T is a torus and therefore commutative,

$T \subset C_G(T)$, and so we have shown that $C_G(T) = T$. In particular, applying Proposition 3.2.1.2 shows that T is regular.

We wish to show that T is a maximal torus. To do so, we assume that $T \subset T'$ for some maximal torus T' of G. Now, since G is reductive and T is regular, we can apply Corollary 3.4.2.1(2), which says

$$T = C_G(T) = T'$$

from which we can conclude that T is maximal.

5.2.3 Borel Subgroup

Let $B = SL(l+1, K) \cap T(l+1, K)$. Then B is solvable, since it is a subgroup of the solvable group $T(l+1, K)$. It is clear that B is connected, since it is generated by T and the subgroups U_{ij}, which consist of matrices with 1's on the diagonal and arbitrary elements in the $(i,j)^{\text{th}}$ entry. Furthermore, it is not difficult to show that $\dim B = (l+1)(l+2)/2 - 1$

Since $G = SL(l+1, K)$ is reductive, we can apply Corollary 3.4.5.2. In particular, if $B' \subset G$ is a Borel subgroup of G, then

$$\dim(B') = \frac{1}{2}(\dim G + \operatorname{rank}(G)) = \frac{1}{2}\big((l+1)^2 - 1 + l\big) = (l^2+3l)/2 = (l+1)(l+2)/2 - 1$$

We see, then, that B as defined above is a connected, solvable group of maximal dimension, and so is a Borel subgroup.

5.2.4 Root System

It is a standard calculation to show that, for $G = SL(l+1, K)$, the algebra $\mathfrak{g} = \mathscr{L}(G)$ consists of matrices of trace 0. See, for example, §1.2 of [4]. Moreover,

$$\mathscr{L}(T) = \mathfrak{t} = \left\{ \begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_l & \\ & & & -(a_1 + \cdots + a_l) \end{pmatrix} \middle| \, a_i \in K \right\}$$

Now, let $\mathsf{x} \in \operatorname{Mat}(l+1, K)$ with $\operatorname{tr}(\mathsf{x}) = 0$. Then, for $t = \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \in T$, where $t_{l+1} = \prod_{i=1}^{l} t_i^{-1}$, we have

$$\left(t\mathsf{x}t^{-1}\right)_{rs} = (t_r/t_s)\mathsf{x}_{rs}$$

for $1 \leq r, s \leq l+1$. It follows that, if $\mathsf{x} = E_{i,j}$, that is, the matrix with 1 in the $(i,j)^{\text{th}}$ entry and 0's elsewhere, we have

$$t\mathsf{x}t^{-1} = (t_i/t_j)\mathsf{x} \tag{5.5}$$

for arbitrary $t \in T$. This tells us that, for $1 \leq i, j \leq l+1$, with $i \neq j$, we have $E_{i,j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i/t_j \tag{5.6}$$

Since $SL(l+1, K)$ is reductive, $\mathfrak{g}_\alpha$ is of dimension 1, and so

$$\mathfrak{g}_\alpha = \{cE_{i,j} \mid c \in K\}$$

By the fact that G is reductive, we can use Theorem 3.4.3.2(5) again, to get

$$\mathfrak{g} = \mathfrak{t} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

By comparison of dimensions, the $l(l+1)$ roots of the form given in equation (5.6) comprise all the roots of $SL(l+1, K)$.

We finally note that, due to the fact that $t_1 \cdots t_{l+1} = 1$, if $j = l+1$, then

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i/t_{l+1} = t_1 \cdots t_i^2 \cdots t_l$$

On the other hand, if $i = l+1$, then

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_{l+1}/t_j = t_1^{-1} \cdots t_j^{-2} \cdots t_l^{-1}$$

It therefore follows that the roots of $SL(l+1, K)$ are the $l(l+1)$ characters

$$\operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i, j \leq l,\ i \neq j \tag{5.7}$$

$$\operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_1 \cdots t_i^2 \cdots t_l, \quad 1 \leq i \leq l \tag{5.8}$$

$$\operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_1^{-1} \cdots t_j^{-2} \cdots t_l^{-1}, \quad 1 \leq j \leq l \tag{5.9}$$

5.2.5 Simple Roots

Let $1 \leq i \leq l$, and define a root α_i as follows

$$\alpha_i : \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i/t_{i+1}$$

and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. We wish to show that Δ is a base for the root system Φ, that is, we need to show that each root $\alpha \in \Phi$ can be written in the form $\sum c_i\alpha_i$ where all the c_i's have the same sign. If we let $\alpha \in \Phi$ be the root

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i/t_j$$

where $1 \leq i < j \leq l+1$, then

$$\alpha = \alpha_i + \cdots + \alpha_{j-1}$$

On the other hand, if $1 \leq j < i \leq l+1$, then

$$\alpha = -\alpha_j - \cdots - \alpha_{i-1}$$

We have shown that each root $\alpha \in \Phi$ can be written in the form $\sum c_i\alpha_i$ where all the c_i's have the same sign, and so Δ is indeed a base for Φ. The associated set of positive roots consists of the following $l(l+1)/2$ distinct roots

$$\operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i < j \leq l+1 \tag{5.10}$$

We return to the Borel subgroup $B = T(l+1,K) \cap SL(l+1,K)$ which we considered above. Firstly, since $B \subset T(l+1,K)$, it is immediate that $\mathscr{L}(B) \subset \mathfrak{t}(l+1,K) \cap \mathfrak{sl}(l+1,K)$. Certainly $\mathfrak{t}(l+1,K) \cap \mathfrak{sl}(l+1,K)$ consists of upper triangular matrices of trace 0. It is immediate that the dimension of this algebra is $(l+1)(l+2)/2 - 1 = l^2 + 3l$. On the other hand, we saw above that $\dim(\mathscr{L}(B)) = \dim B = l^2 + 3l$, and so we conclude that

$$\mathscr{L}(B) = \mathfrak{t}(l+1,K) \cap \mathfrak{sl}(l+1,K)$$

That is, $\mathscr{L}(B)$ consists of upper triangular matrices of trace 0. But the elements of the root spaces $\mathfrak{g}_\alpha$ where α is one of the positive roots given in (5.10) above are all upper triangular and of trace 0. That is, $\mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha = \mathscr{L}(B)$, and so the base Δ which we have defined here is equal to $\Delta(B)$ for $B = SL(l+1,K) \cap T(l+1,K)$.

5.2.6 Weyl Group

To calculate $W(G,T)$, with $T = D(l+1,K) \cap G$, we first calculate the normaliser $N_G(T)$.

It is easy to show that $N_{GL(l+1,K)}\big(D(l+1,K)\big) \cap G \subset N_G(T)$, which means that all the monomial matrices of determinant one lie in $N_G(T)$. On the other hand, suppose $x \in N_G(T)$. Then, for all $t \in T$, we have $xtx^{-1} = s$ for some $s \in T$. That is, $x = s^{-1}xt$, and so

$$x_{ij} = (t_j/s_i)x_{ij}$$

Suppose $x_{ij} \neq 0$. Then clearly, for the above equality to hold, we must have $s_i = t_j$. Suppose now k is such that $x_{ik} \neq 0$. Then, likewise, $s_i = t_k$. But then $t_i = t_k$, and this must hold for arbitrary $t \in T$, from which we must conclude $j = k$, which is to say that each row of x contains at most 1 nonzero element. But since x is invertible, each row of x must contain at least 1 nonzero element, and so each row of x has precisely 1 nonzero element. The conclusion is that x is monomial. We have shown, then, that $N_G(T)$ consists of the monomial matrices of determinant one. That is, $N_G(T) = N_{GL(l+1,K)}\big(D(l+1,K)\big) \cap G$.

Now, if $x \in N_G(T)$, then $x = p_\sigma t$ for some particular $\sigma \in \mathfrak{S}_{l+1}$ and some $t \in D(l+1,K)$. Note, however, that if $\operatorname{sgn}(\sigma) = -1$ then $\det t = -1$, and so $t \notin T$.

$$\begin{aligned} w : W &\longrightarrow \mathfrak{S}_{l+1} \\ (p_\sigma t)T &\longmapsto \sigma \end{aligned}$$

We claim firstly that this map is well defined. Indeed, if $xT = yT$ for some $x, y \in N_G(T)$, where $x = p_\sigma t$ and $y = p_\tau s$, for $s, t \in D(l+1,K)$, then

$$xy^{-1} = p_\sigma ts^{-1}p_{\tau^{-1}} = p_\sigma p_{\tau^{-1}}t' \in D(l+1,K)$$

where the final equality is because monomial matrices normalise diagonal matrices, as shown above in §5.1.5. This then implies $\sigma = \tau$, and so $w(xT) = w(yT)$.

We need to show now that w is a group homomorphism, so suppose again that $x = p_\sigma t$ and $y = p_\tau s$, for $s, t \in D(l+1, K)$, and consider $w\big((xT)(yT)\big)$:

$$w\big((xT)(yT)\big) = w(p_\sigma t p_\tau s) = w(p_\sigma p_\tau t' s) = w(p_{\sigma\tau} t' s) = \sigma\tau = w(xT)w(yT)$$

where the second equality again follows from the fact that monomial matrices normalise diagonal matrices. The homomorphism w is clearly injective, since if $w(xT) = 1$, then $x = p_e t = t$, and so $xT = T$. On the other hand, if $\sigma \in \mathfrak{S}_{l+1}$, then, for

$$t = \begin{pmatrix} \operatorname{sgn}(\sigma) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

the matrix $x = p_\sigma t$ is monomial and of determinant one, and so lies in $N_G(T)$, and, moreover, $w(xT) = w(p_\sigma tT) = \sigma$, and so w is surjective. We have shown, then, that w is a group isomorphism, and therefore $W \cong \mathfrak{S}_{l+1}$.

5.2.7 Dynkin Diagram

Recall from A.7 of [6] that a *Dynkin diagram* of an (irreducible) root system is constructed as follows: We draw a node for each simple root, and join the nodes α_i and α_j by $\langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle$ bonds. This number is in fact a non-negative integer, called the *Cartan integer* between α_i and α_j, so the construction makes sense. If the Cartan integer is nonzero, then we also draw an arrow from the α_i node to the α_j node if and only if the ratio $\langle\alpha_i, \alpha_j\rangle\big(\langle\alpha_j, \alpha_i\rangle\big)^{-1}$ is strictly greater than 1. This corresponds to putting an arrow pointing in the direction of the shorter of the two roots, as the integer $\langle\alpha_i, \alpha_j\rangle$ is inversely proportional to the length of α_j.

We use the base Δ as defined in §5.2.5 to compute the Cartan integers for the root system. To do so we first wish to compute the simple reflections $\sigma_i = \sigma_{\alpha_i}$ explicitly. Thereafter, we use the formula, given in A.3 of [6]:

$$\langle\beta, \alpha\rangle\alpha = \beta - \sigma_\alpha(\beta) \tag{5.11}$$

In order to compute the σ_i for each $i \in \Delta$, we make use of Lemma 3.5.2.4, which says that σ_i is precisely the reflection which sends α_i to $-\alpha_i$ and which sends the remaining simple reflections to the set $\Phi^+ - \{\alpha_i\}$. Our strategy throughout this chapter, then, will be to find reflections which behave in this way, and then assert the fact of Lemma 3.5.2.4.

Consider, then, $\alpha_i \in \Delta$, for $1 \leq i \leq l$, defined according to the formula given in §5.2.5, namely

$$\alpha_i : \operatorname{diag}(t_1, \ldots, t_l, t_{l+1}) \longmapsto t_i/t_{i+1}$$

The appropriate candidate for σ_i is the reflection which corresponds to $(i, i+1)$ under the isomorphism $W \cong \mathfrak{S}_l$. This permutation acts on t by swapping the

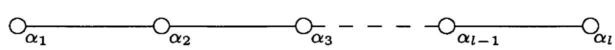
i^{th} and $(i+1)^{\text{th}}$ diagonal entries. So, with the help of the formulae given in §5.2.5, it is easy to calculate the following:

$$\sigma_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } j = i \\ \alpha_{i-1} + \alpha_i & \text{if } j = i-1 \\ \alpha_i + \alpha_{i+1} & \text{if } j = i+1 \\ \alpha_j & \text{otherwise} \end{cases}$$

where of course we ignore the case $j = i-1$ if $i = 1$ and likewise the case $j = i+1$ if $i = l$. Notice that this reflection σ_i inverts α_i and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_i\}$. Therefore, by Lemma 3.5.2.4, this σ_i is indeed the simple reflection corresponding to α_i. Plugging all this into equation (5.11),

$$\langle \alpha_j, \alpha_i \rangle \alpha_i = \alpha_j - \sigma_i(\alpha_j) = \begin{cases} 2\alpha_i & \text{if } j = i \\ -\alpha_i & \text{if } j = i-1 \\ -\alpha_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

This information gives us the Cartan integers, which we can now encode in the Dynkin diagram for $G = SL(l+1, K)$:



5.3 The Symplectic Group

The symplectic group is defined to be the group $Sp(2l, K)$, which consists of all matrices $x \in GL(2l, K)$ satisfying

$$^t x s x = s \tag{5.12}$$

where

$$s = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 0 & & 0 & 1 \\ 0 & & 1 & 0 \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & 0 & 0 \end{pmatrix}$$

This is a group, since it is easily shown to be closed under multiplication and taking of inverses. Since equation (5.12) imposes polynomial conditions on x, $Sp(2l, K)$ is closed in $GL(2l, K)$. We also saw in Proposition 1.3.2.8 that G is connected.

It is a standard calculation to show that $\mathscr{L}(G) = \mathfrak{g}$ consists of matrices of the form

$$\mathsf{x} = \begin{pmatrix} a & b \\ c & -J^t aJ \end{pmatrix} \tag{5.13}$$

where $a, b, c \in \mathrm{Mat}(l, K)$, and ${}^t b = JbJ$, and ${}^t c = JcJ$, which amounts to b and c being symmetric about the skew diagonal. See, for example, §1.8 of [4] or §8.13.3 of [3]. From this we also see that $\dim G = \dim(\mathfrak{g}) = l^2 + l(l+1) = 2l^2 + l$.

5.3.1 Maximal Torus

Consider the set $T = Sp(2l, K) \cap D(2l, K)$. Let $x \in T$. In fact, if we rewrite x as

$$x = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}$$

where $y, z \in D(l, K)$, then

$${}^t xsx = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & yJz \\ -zJy & 0 \end{pmatrix}$$

Since $x \in Sp(2l, K)$, we have ${}^t xsx = s$, or $J = yJz = zJy$. But it is easy to show

$$(yJz)_{ij} = y_{ii} J_{ij} z_{jj}$$

and so, if $i + j = l + 1$, then $z_{jj} = 1/y_{ii}$. This tells us that x must be of the form

$$x = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_l & & & \\ & & & t_l^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \tag{5.14}$$

On the other hand, if a matrix x has the form described in (5.14), then it is easy show that ${}^t xsx = s$, and so x is symplectic. Now, clearly the set T consists of matrices of the form described in (5.14), and so T is isomorphic to $D(l, K)$, and so is a torus.

We wish to compute the group $C_{G'}(T)$ where $G' = GL(2l, K)$. Suppose, then, that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C_{G'}(T) \subset G'$, and let x be an arbitrary element of T, say $x = \begin{pmatrix} t & 0 \\ 0 & Jt^{-1}J \end{pmatrix}$ where $t = \mathrm{diag}(t_1, \ldots, t_l) \in D(l, K)$. Then

$$\begin{aligned} xgx^{-1} &= \begin{pmatrix} t & 0 \\ 0 & Jt^{-1}J \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & JtJ \end{pmatrix} \\ &= \begin{pmatrix} tat^{-1} & tb(JtJ) \\ (Jt^{-1}J)ct^{-1} & (Jt^{-1}J)d(JtJ) \end{pmatrix} \end{aligned}$$

and this must be equal to g. In particular, a is such that $tat^{-1} = a$. But we already saw in §5.2.2 that this condition forces $a \in D(l, K)$. Similarly,

$$\left((Jt^{-1}J)d(JtJ)\right)_{ij} = (t_{l+1-i}/t_{l+1-j})d_{ij}$$

must be equal to d_{ij} for arbitrary t, which likewise forces $d \in D(l, K)$. Now

$$\left(tb(JtJ)\right)_{ij} = (t_i t_{l+1-j})b_{ij}$$

must be equal to b_{ij} for arbitrary t. But if b_{ij} is nonzero, we can always find a t with $t_i \neq t_{l+1-j}^{-1}$, which would contradict this equality. Therefore, $b = 0$. Similarly,

$$\left((Jt^{-1}J)ct^{-1}\right)_{ij} = (t_{l+1-i}^{-1}t_j^{-1})c_{ij}$$

must be equal to c_{ij} for arbitrary t. But if c_{ij} is nonzero, we can always find a t with $t_j \neq t_{l+1-i}^{-1}$, which would likewise contradict this equality. Again, we must conclude that $c = 0$. We have shown, then, that if $x \in C_{G'}(T)$, then $x \in D(2l, K)$. Of course, $D(2l, K)$ is commutative, so the reverse inclusion holds, too, and thus we get $C_{G'}(T) = D(2l, K)$.

From this we observe that $C_G(T) = G \cap C_{G'}(T) = G \cap D(2l, K) = T$. Suppose now that T' is a maximal torus, containing T. It is immediate that T' is also a maximal torus of $C_G(T)$, and so we get the inclusions

$$T \subset T' \subset C_{G'}(T) = T$$

from which we conclude that $T = T'$, that is, T is a maximal torus.

5.3.2 Root System

We continue with the notation $G = Sp(2l, K)$, and, as above, $T = G \cap D(2l, K)$. Then certainly

$$\mathscr{L}(T) \subset \mathfrak{g} \cap \mathfrak{d}(2l, K) = \left\{ \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_l & & & \\ & & & -a_l & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix} \right\}$$

where $a_i \in K$ for $1 \leq i \leq l$. This last equality is simply by directly computing the intersection based on (5.13). But $\dim(\mathscr{L}(T)) = \dim T = l$, and this is clearly equal to the dimension of $\mathfrak{g} \cap \mathfrak{d}(2l, K)$, and so the above inclusion is equality. Moreover, Proposition 2.3.1.9 says that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathscr{L}(C_G(T))$, but this latter algebra is equal to $\mathscr{L}(T)$ since $T = C_G(T)$. Altogether, then, we have

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathscr{L}(T) = \mathfrak{g} \cap \mathfrak{d}(2l, K) \tag{5.15}$$

That is, the centraliser of the T-action on $\mathfrak{g}$ is equal to $\mathfrak{g} \cap \mathfrak{d}(2l, K)$.

Label the standard basis of K^{2l} by $f_{\pm 1}, \dots, f_{\pm l}$, such that, for $1 \le i \le l$, $f_i = e_i$ and $f_{-i} = e_{2l+1-i}$. Let $F_{i,j}$ be the matrix which sends f_j to f_i, where $-l \le i, j \le l$. That is, $F_{i,j}$ has zero entries everywhere except the $(i,j)^{\text{th}}$ entry, which is 1.

Firstly, suppose $1 \le i, j \le l$, with $i \ne j$, and define the matrix $X_{\epsilon_i - \epsilon_j} = F_{i,j} - F_{-j,-i}$. That is,

$$X_{\epsilon_i-\epsilon_j} = \left(\begin{array}{c|c} E_{i,j} & 0 \\ \hline 0 & -E_{l+1-j,l+1-i} \end{array}\right)$$

where here the $E_{r,s}$ matrices are the $l \times l$ elementary matrices. Note further that $E_{l+1-j,l+1-i} = J^t E_{i,j} J$, and so comparison with (5.13) shows $X_{\epsilon_i-\epsilon_j} \in \mathfrak{g}$.

Now, for any matrix $t = \operatorname{diag}(t_1, \dots, t_l, t_l^{-1}, \dots t_1^{-1}) \in T$, and writing $t' = \operatorname{diag}(t_1, \dots, t_l)$, we have

$$\begin{aligned} tX_{\epsilon_i-\epsilon_j}t^{-1} &= \left(\begin{array}{c|c} t' & 0 \\ \hline 0 & Jt'^{-1}J \end{array}\right) \left(\begin{array}{c|c} E_{i,j} & 0 \\ \hline 0 & E_{l+1-j,l+1-i} \end{array}\right) \left(\begin{array}{c|c} t'^{-1} & 0 \\ \hline 0 & Jt'J \end{array}\right) \\ &= \left(\begin{array}{c|c} t'E_{i,j}t'^{-1} & 0 \\ \hline 0 & -Jt'^{-1}JE_{l+1-j,l+1-i}Jt'J \end{array}\right) \end{aligned}$$

Now, for $1 \le r, s \le l$, we have

$$\left(t'E_{i,j}t'^{-1}\right)_{rs} = t_r/t_s \left(E_{i,j}\right)_{rs} = \begin{cases} t_i/t_j & \text{if } r = i \text{ and } s = j, \\ 0 & \text{otherwise,} \end{cases}$$

and so

$$t'E_{i,j}t'^{-1} = \frac{t_i}{t_j}E_{i,j} \tag{5.16}$$

A very similar argument shows

$$Jt'^{-1}JE_{l+1-j,l+1-i}Jt'J = \frac{t_i}{t_j}E_{l+1-j,l+1-i} \tag{5.17}$$

Together, equations (5.16) and (5.17) tell us that

$$tX_{\epsilon_i-\epsilon_j}t^{-1} = \frac{t_i}{t_j}X_{\epsilon_i-\epsilon_j}$$

Therefore, $X_{\epsilon_i-\epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \operatorname{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i/t_j$$

Now, let $1 \le i < j \le l$, and define a matrix $X_{\epsilon_i+\epsilon_j} = F_{i,-j} + F_{j,-i}$. That is,

$$X_{\epsilon_i+\epsilon_j} = \left(\begin{array}{c|c} 0 & E_{i,l+1-j} + E_{j,l+1-i} \\ \hline 0 & 0 \end{array}\right)$$

Since this upper right component is symmetric about its skew diagonal, (5.13) tells us that $X_{\epsilon_i+\epsilon_j} \in \mathfrak{g}$. Now, for any matrix $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots t_1^{-1}) \in T$, elementary calculations like those of (5.16) and (5.17) show that

$$tX_{\epsilon_i+\epsilon_j}t^{-1} = t_i t_j X_{\epsilon_i+\epsilon_j}$$

so $X_{\epsilon_i+\epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i t_j$$

Now, let $1 \leq i < j \leq l$, and define a matrix $X_{-\epsilon_i-\epsilon_j} = F_{-j,i} + F_{-i,j}$. That is,

$$X_{-\epsilon_i-\epsilon_j} = \left(\begin{array}{c|c} 0 & 0 \\ \hline E_{l+1-j,i} + E_{l+1-i,j} & 0 \end{array}\right)$$

Since this lower left component is symmetric about its skew diagonal, (5.13) tells us that $X_{-\epsilon_i-\epsilon_j} \in \mathfrak{g}$. Now, for any matrix $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots t_1^{-1}) \in T$,

$$tX_{-\epsilon_i-\epsilon_j}t^{-1} = \frac{1}{t_i t_j} X_{-\epsilon_i-\epsilon_j}$$

so $X_{-\epsilon_i-\epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}$$

Now let $1 \leq i \leq l$, and define a matrix $X_{2\epsilon_i} = F_{i,-i}$. That is, with 1 in the $(i, 2l+1-i)^{\text{th}}$ entry, and 0s everywhere else. Since the nonzero component lies on the skew diagonal, comparison with (5.13) tells us that $X_{2\epsilon_i} \in \mathfrak{g}$. Now, for any matrix $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots t_1^{-1}) \in T$, and writing $t' = \mathrm{diag}(t_1, \dots, t_l)$, we have

$$tX_{2\epsilon_i}t^{-1} = t_i^2 X_{2\epsilon_i}$$

so $X_{2\epsilon_i} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i^2$$

Finally, let $1 \leq i \leq l$, and define a matrix $X_{-2\epsilon_i} = F_{-i,i}$. That is, with 1 in the $(2l+1-i, i)^{\text{th}}$ entry, and 0s everywhere else. Again, the nonzero component lies on the skew diagonal, and so comparison with (5.13) tells us that $X_{2\epsilon_i} \in \mathfrak{g}$. Now, for any matrix $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots t_1^{-1}) \in T$, and writing $t' = \mathrm{diag}(t_1, \dots, t_l)$, we have

$$tX_{-2\epsilon_i}t^{-1} = t_i^{-2} X_{-2\epsilon_i}$$

so $X_{-2\epsilon_i} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i^{-2}$$

We have therefore shown that the following $2l^2$ distinct characters are roots of $G = Sp(2l, K)$:

$$\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i, j \leq l, \, i \neq j \tag{5.18}$$

$$\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j, \quad 1 \leq i, j \leq l \tag{5.19}$$

$$\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}, \quad 1 \leq i, j \leq l \tag{5.20}$$

We can now use Lemma 3.2.3.2, which says $\mathrm{Card}(\Phi) \leq \dim(G) - \dim(C_G(T)) = 2l^2 + l - l$, and so the above list of roots exhausts the elements of Φ. In turn we can conclude that $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$.

By a direct comparison between the spaces $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ computed above, and (5.13), it is apparent that we have produced a decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{5.21}$$

5.3.3 Simple Roots

Suppose $l = 1$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(2, K)$. Then

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad - bc \\ bc - ad & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore we see that $Sp(2, K) = SL(2, K)$, and so we have dealt with the $l = 1$ case in §5.2. We therefore turn our attention to the cases $l \geq 2$.

Let $1 \leq i \leq l - 1$, and define a root α_i of type (5.18) as follows

$$\alpha_i : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i / t_{i+1}$$

and define a root α_l of type (5.19) as follows:

$$\alpha_l : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_l^2$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. We wish to show that Δ is a set of simple roots, which means we have to be able to express all roots α G in the form $\sum c_i \alpha_i$, where the c_i's are scalars of like sign. Suppose firstly that $\alpha \in \Phi$ is a root of the form given above in (5.18), say

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i / t_j$$

where $i \neq j$. If $i < j$, then

$$\alpha = \alpha_i + \cdots + \alpha_{j-1}$$

On the other hand, if $i > j$, then

$$\alpha = -\alpha_j - \cdots - \alpha_{i-1}$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.19),

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i t_j$$

If $1 \leq i, j < l-1$ and $i \neq j$, then

$$\alpha = (\alpha_i + \dots + \alpha_{l-1}) + (\alpha_j + \dots + \alpha_{l-1}) + \alpha_l$$

On the other hand, if $1 \leq i, j < l-1$ and $i = j$, then

$$\alpha = 2\alpha_i + \dots + 2\alpha_{l-1} + \alpha_l$$

Suppose now that $1 \leq i < l$, and $j = l$. Then

$$\alpha = \alpha_i + \dots + \alpha_{l-1} + \alpha_l$$

Finally, suppose $i = j = l$, then, of course $\alpha = \alpha_l$. This deals with all the cases where α is of the form (5.19). The only case left is when α is a root of the form given above in (5.20),

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}$$

and $1 \leq i, j \leq l$. Suppose firstly that $1 \leq i, j < l$, and $i \neq j$. Then

$$\alpha = -(\alpha_i + \dots + \alpha_{l-1}) - (\alpha_j + \dots + \alpha_{l-1}) - \alpha_l$$

On the other hand, if $1 \leq i, j < l-1$ and $i = j$, then

$$\alpha = -(2\alpha_i + \dots + 2\alpha_{l-1} + \alpha_l)$$

Suppose now that $1 \leq i < l$, and $j = l$. Then

$$\alpha = -(\alpha_i + \dots + \alpha_{l-1} + \alpha_l)$$

Finally, suppose $i = j = l$, then, of course $\alpha = -\alpha_l$.

We have now shown that Δ is a base for Φ. The associated set of positive roots consists of the following l^2 distinct roots:

$$\begin{aligned} &\mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i < j \leq l \\ &\mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i t_j, \quad 1 \leq i, j \leq l \end{aligned} \tag{5.22}$$

5.3.4 Radical and Unipotent Radical

We begin this section by explicitly calculating some Lie brackets of elements of the Lie algebra $\mathfrak{g}$. In particular, let $\alpha \in \Phi$, and take a nonzero element $\mathsf{x} \in \mathfrak{g}_\alpha$. We see from our computation of the elements of Φ in §5.3.2 above that $\Phi = -\Phi$, and so $-\alpha$ is also a root of G. Take a nonzero element $\mathsf{y} \in \mathfrak{g}_{-\alpha}$. We wish to show that the bracket $[\mathsf{x}, \mathsf{y}]$ is nonzero for each $\alpha \in \Phi$.

Suppose firstly that α is of the form given in (5.18). Then, since $\dim \mathfrak{g}_\alpha = 1$, using the notation of §5.3.2, x is a nonzero multiple of $X_{\epsilon_i - \epsilon_j} = F_{i,j} - F_{-j,-i}$, with $i \neq j$, and y is a nonzero multiple of $X_{\epsilon_j - \epsilon_i}$. By direct calculation, we have

$$[X_{\epsilon_i-\epsilon_j}, X_{\epsilon_j-\epsilon_i}] = F_{i,i} - F_{j,j} + F_{-j,-j} - F_{-i,-i}$$

which is not zero, since $i \neq j$. Now $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of $[X_{\epsilon_i-\epsilon_j}, X_{\epsilon_j-\epsilon_i}]$, and so is nonzero.

Suppose now that α is of the form given in (5.19). Then again x is necessarily a nonzero multiple of $X_{\epsilon_i+\epsilon_j} = F_{i,-j} + F_{j,-i}$, since since $\dim \mathfrak{g}_\alpha = 1$, and similarly y is a nonzero multiple of $X_{-\epsilon_i,-\epsilon_j} = F_{-j,i} + F_{-i,j}$. By direct calculation,

$$[X_{\epsilon_i+\epsilon_j}, X_{-\epsilon_j-\epsilon_i}] = F_{i,i} + F_{j,j} - F_{-j,-j} - F_{-i,-i}$$

which is nonzero for all i, j. Since $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of the above expression, it too is nonzero. Finally note that the case where α is of the form given in (5.20) is a consequence of the above calculation, since a root of this form is inverse to one of the form (5.19). We have therefore shown that if $\mathsf{x} \in \mathfrak{g}_\alpha$ is not zero, and $\mathsf{y} \in \mathfrak{g}_{-\alpha}$ is not zero, then $[\mathsf{x}, \mathsf{y}] \neq 0$.

We now wish to consider $R_u(G)$, which we will denote by U. Since U is a closed, normal subgroup of G, then $\mathscr{L}(U) = \mathfrak{u}$ is an ideal in $\mathfrak{g}$, by Corollary 10.4A of [6]. Moreover, since U is unipotent, Corollary 2.2.4.4 tells us that $\mathfrak{u}$ consists of nilpotent elements. Consider an element $\mathsf{x} \in \mathfrak{u}$. Since $\mathsf{x} \in \mathfrak{g}$, we can use (5.21) to express this element as $\mathsf{x} = \mathsf{x}_0 + \sum \mathsf{x}_\alpha$ where $\mathsf{x}_0 \in \mathfrak{t}$ and $\mathsf{x}_\alpha \in \mathfrak{g}_\alpha$. Now, U is normal, so is stable under $\mathrm{Int}x$ for all $x \in G$. Therefore $\mathfrak{u}$ is stable under $\mathrm{Ad}x$ for all $x \in G$. In particular, $t.\mathsf{x} \in \mathfrak{u}$ for all $t \in T$. On the other hand,

$$t.\mathsf{x} = t.\mathsf{x}_0 + \sum t.\mathsf{x}_\alpha = \mathsf{x}_0 + \sum \alpha(t)\mathsf{x}_\alpha$$

Indeed, we can repeat this action i times to get

$$t^i.\mathsf{x} = \mathsf{x}_0 + \sum \alpha^i(t)\mathsf{x}_\alpha$$

We saw in the proof of Proposition 2.3.1.8 that we can always find an element $t \in T$ such that, given any roots $\alpha, \beta \in \Phi$, with $\alpha \neq \beta$, then $\alpha(t) \neq \beta(t)$. Indeed, we can even choose t such that $\alpha^i(t) \neq \beta^i(t)$ for each $i > 0$. We construct an $n \times n$ matrix from these powers of eigenvalues, where $n - 1 = \mathrm{Card}(\Phi)$:

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & \alpha(t) & \beta(t) & \cdots \\ \vdots & & \ddots & \\ 1 & \alpha^{n-1}(t) & \beta^{n-1}(t) & \cdots \end{pmatrix}$$

Then, from what we have just observed, if construct the column vector $b = {}^t(\mathsf{x}_0, \mathsf{x}_\alpha, \mathsf{x}_\beta, \ldots)$, then

$$Ab \in \mathfrak{u}^n$$

But, by construction, A is invertible, and so $b \in \mathfrak{u}^n$, and in particular, $\mathsf{x}_0 \in \mathfrak{u}$, and $\mathsf{x}_\alpha \in \mathfrak{u}$ for all $\alpha \in \Phi$.

In the first place, we have shown that $\mathsf{x}_0 \in \mathfrak{u} \cap \mathfrak{t}$. But $\mathfrak{u}$ consists of nilpotent elements, and $\mathfrak{t}$ consists of semisimple elements, and so $\mathsf{x}_0 = 0$. On the other hand, we have shown $\mathsf{x}_\alpha \in \mathfrak{u} \cap \mathfrak{g}_\alpha$. Suppose this element is nonzero. By examining the list of roots calculated in §5.3.2, we know that $-\alpha \in \Phi$, so select a nonzero element $\mathsf{y} \in \mathfrak{g}_{-\alpha}$. By Lemma 3.4.3.1, we know that $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{g}_0 = \mathfrak{t}$. On the other hand, we observed above that $\mathfrak{u}$ is an ideal, and so $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{u}$ since $\mathsf{x}_\alpha \in \mathfrak{u}$. We have shown, then, that $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{u} \cap \mathfrak{t}$, and so, being both semisimple and nilpotent, we are forced to conclude that $[\mathsf{x}_\alpha, \mathsf{y}] = 0$. But this is a contradiction of the argument which began this section, and so it follows that $\mathsf{x}_\alpha = 0$ for all $\alpha \in \Phi$.

We have therefore shown that $\mathfrak{u} = \{0\}$. In particular, $0 = \dim \mathfrak{u} = \dim U$. But U is connected, and so must be trivial. Therefore G is reductive.

In fact, we can say more. By Theorem 3.4.3.2(6), $Z(G)^\circ = \left(\bigcap_{\alpha \in \Phi} T_\alpha\right)^\circ$. Suppose, then, that $t = \operatorname{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \in T$ is such that $\alpha(t) = 1$ for all $\alpha \in \Phi$. In particular, $t_i/t_j = 1$ and $t_i t_j = 1$ for all i, j. Therefore $t = \pm I$ where I is the identity matrix. It follows that the connected component of $Z(G)$ is trivial, but since G is reductive, we can apply Lemma 3.1.1.9 which tells us $R(G) = Z(G)^\circ$, and so we conclude $R(G)$ is trivial, and therefore G is semisimple.

5.3.5 Borel Subgroup

Let $g \in T(2l, K)$, be of the form

$$g = \begin{pmatrix} x & y \\ 0 & x' \end{pmatrix}$$

Then it is not difficult to see g is symplectic, that is, satisfies equation (5.12), if and only if

1. ${}^t x J x' = J$ and
2. ${}^t x' J y = {}^t y J x'$.

Note that (2) implies that ${}^t x' J y$ is symmetric. Now consider a set B consisting of matrices of the form

$$g = \begin{pmatrix} x & xJz \\ 0 & J {}^t x^{-1} J \end{pmatrix}$$

where x is in $T(l, K)$, and z is an arbitrary $l \times l$ symmetric matrix. Then, we claim, $B = T(2l, K) \cap Sp(2l, K)$. Indeed, given $g \in B$, it is clear enough that $x \in T(2l, K)$. Secondly, it is easy to check that g satisfies both (1) and (2) above.

On the other hand, suppose $g \in T(2l, K) \cap Sp(2l, K)$. Then, since (1) and (2) hold for g, it is easy to see that $g \in B$, from which we can conclude $B = T(2l, K) \cap Sp(2l, K)$.

Now, certainly B is solvable, since it is a subgroup of the solvable group $T(2l, K)$. On the other hand, by construction B is isomorphic to the variety $T(l, K) \times X_l$, where X_l consists of symmetric $l \times l$ matrices. Note that X_l is an irreducible variety, since it is isomorphic to $\mathbf{A}^{n(n+1)/2}$, and since $T(l, K)$ is connected, it follows that B is connected. Moreoever, this isomorphism of varieties implies

$$\dim B = \dim T(l, K) + \dim X_l = l(l+1)/2 + l(l+1)/2 = l^2 + l$$

We wish to show that B is a Borel subgroup. Since it is solvable and connected, it lies inside some Borel subgroup B'. G is reductive, so we can apply Theorem 3.4.3.2, which, amongst many other things, says

$$\dim B' = \dim G - \frac{1}{2}\text{Card}\Phi = 2l^2 + l - l^2 = l^2 + l$$

Therefore $\dim B = \dim B'$, which means $B = B'$, and so B is a Borel subgroup.

Now it is clear that, since $B \subset T(2l, K)$, we have $\mathscr{L}(B) \subset \mathfrak{t}(2l, K)$ and since $B \subset Sp(2l, K)$, we have $\mathscr{L}(B) \subset \mathfrak{sp}(2l, K)$, which means $\mathscr{L}(B) \subset \mathfrak{t}(2l, K) \cap \mathfrak{sp}(2l, K)$. Furthermore, by direct observation of (5.13), we can see that

$$\mathfrak{t}(2l, K) \cap \mathfrak{sp}(2l, K) = \left\{ \begin{pmatrix} a & b \\ c & -J^t a J \end{pmatrix} \middle| \, a \in \mathfrak{t}(l, K), b = J^t b J \right\}$$

which in turn gives

$$\dim \big(\mathfrak{t}(2l, K) \cap \mathfrak{sp}(2l, K)\big) = \frac{1}{2}l(l+1) + \frac{1}{2}l(l+1) = l + l^2$$

We conclude, therefore that $\mathscr{L}(B) = \mathfrak{t}(2l, K) \cap \mathfrak{sp}(2l, K)$. It is now obvious that the roots of B are precisely the positive roots described in (5.22), and so the base given in §5.3.3 is equal to $\Delta(B)$ for $B = T(2l, K) \cap G$.

5.3.6 Weyl Group

We first calculate the $N_G(T)$. It is immediate that $N_{GL(2l,K)}\big(D(2l, K)\big) \cap G \subset N_G(T)$. On the other hand, suppose $x \in N_G(T)$. Then, for all $t \in T$, we have $xtx^{-1} = t'$ for some $t' \in T$. That is, $x = t'^{-1}xt$, and so

$$x_{ij} = (t_j / t'_i) x_{ij}$$

Suppose $x_{ij} \neq 0$. Then clearly, for the above equality to hold, we must have $t'_i = t_j$. Suppose now k is such that $x_{ik} \neq 0$. Then, likewise, $t'_i = t_k$. But then $t_i = t_k$, but this must hold for arbitrary $t \in T$, from which we must conclude $j = k$, which is to say that each row of x contains at most 1 nonzero element. But since x is invertible, each row of x must contain at least 1 nonzero element. The conclusion is that each row of x has precisely 1 nonzero element, that is, x is monomial. We have shown, then, that $N_G(T)$ consists of the monomial matrices which lie in G. That is, $N_G(T) = N_{GL(2l,K)}\big(D(2l, K)\big) \cap G$.

We ought to work out precisely which monomial matrices lie in G. For this we again use the easily observable fact that any $n \times n$ monomial matrix x can be written $x = p_\sigma t$ for t diagonal, and $p\sigma$ the permutation matrix corresponding to the permutation $\sigma \in \mathfrak{S}_n$. In particular, we can write our matrix $s = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$ in this form, since s is certainly monomial. Indeed, if we define the permutation $\gamma \in \mathfrak{S}_{2l}$ to be

$$\gamma = (2l, 1)(2l-1, 2)\cdots \tag{5.23}$$

then $s = p_\gamma a$, where a is the diagonal matrix

$$a = \begin{pmatrix} -I_l & 0 \\ 0 & I_l \end{pmatrix}$$

where I_l denotes the $l \times l$ identity matrix.

Suppose now that $x = p_\sigma t$ is a monomial matrix contained in G. Then ${}^t x s x = s$ becomes

$$\begin{aligned} p_\gamma a = {}^t(p_\sigma t)(p_\gamma a)(p_\sigma t) &= (t p_{\sigma^{-1}})(p_\gamma a)(p_\sigma t) \\ &= (p_{\sigma^{-1}} p_\gamma p_\sigma) t' \end{aligned} \tag{5.24}$$

for some diagonal t', where this last equality holds because we saw in §5.1.5 that diagonal matrices are normalised by permutation matrices. An immediate consequence of equation (5.24) is

$$\gamma = \sigma^{-1} \gamma \sigma \tag{5.25}$$

and so $x = p_\sigma t$ is an element of G if and only if σ commutes with γ.

We now turn to the computation of $W = N_G(T)/T$. Firstly, let $\pi \in \mathfrak{S}_l$ be the permutation

$$\pi = (1, l)(2, l-1)(3, l-2)\cdots$$

Then it is easy to see that $J = p_\pi$, the permutation matrix associated to π.

For each $\sigma \in \mathfrak{S}_l$, define a matrix $p(\sigma)$ in $GL(2l, K)$ as follows:

$$p(\sigma) = \begin{pmatrix} p_\sigma & 0 \\ 0 & p_\pi p_\sigma p_\pi \end{pmatrix}$$

Note that $\det p(\sigma) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi\sigma\pi) = \mathrm{sgn}(\sigma)^2 \mathrm{sgn}(\pi)^2 = 1$. We claim further that $p(\sigma) \in G$. Indeed,

$$\begin{aligned} {}^t p(\sigma) s p(\sigma) &= \begin{pmatrix} p_\sigma^{-1} & 0 \\ 0 & p_\pi p_\sigma^{-1} p_\pi \end{pmatrix} \begin{pmatrix} 0 & p_\pi \\ -p_\pi & 0 \end{pmatrix} \begin{pmatrix} p_\sigma & 0 \\ 0 & p_\pi p_\sigma p_\pi \end{pmatrix} \\ &= \begin{pmatrix} 0 & p_\sigma^{-1} p_\pi p_\pi p_\sigma p_\pi \\ -p_\pi p_\sigma^{-1} p_\pi p_\pi p_\sigma & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & p_\pi \\ -p_\pi & 0 \end{pmatrix} \\ &= s \end{aligned}$$

as required. Suppose now that $t = \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \in T$, and denote it by $t = \begin{pmatrix} t' & 0 \\ 0 & t'' \end{pmatrix}$. Then

$$\begin{aligned} p(\sigma)tp(\sigma)^{-1} &= \begin{pmatrix} p_\sigma & 0 \\ 0 & p_\pi p_\sigma p_\pi \end{pmatrix} \begin{pmatrix} t' & 0 \\ 0 & t'' \end{pmatrix} \begin{pmatrix} p_\sigma^{-1} & 0 \\ 0 & p_\pi p_\sigma^{-1} p_\pi \end{pmatrix} \\ &= \begin{pmatrix} p_\sigma t' p_\sigma^{-1} & 0 \\ 0 & p_{\pi\sigma\pi} t'' p_{\pi\sigma\pi}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} s' & 0 \\ 0 & s'' \end{pmatrix} \in T \end{aligned}$$

Where the final equality holds because permutation matrices normalise diagonal matrices, as was shown in §5.1.5. It follows, then, that $p(\sigma) \in N_G(T)$ for all $\sigma \in \mathfrak{S}_l$.

Define a map

$$\begin{aligned} w : \mathfrak{S}_l &\longrightarrow W \\ \sigma &\longmapsto p(\sigma)T \end{aligned}$$

It is easy to see that $p(\sigma)p(\tau) = p(\sigma\tau)$, and so w is a group homomorphism. Moreover, if $w(\sigma) = e$, then $p(\sigma) \in T$, which implies $p_\sigma \in D(l, K)$, which forces $\sigma = e$, and so w is injective.

Recall the basis $\{f_{\pm i}\}_{1 \le i \le l}$ for K^{2l} used above. We define a monomial matrix $n_i \in GL(2l, K)$, for $1 \le i \le l$, in terms of how it acts on these basis elements. In particular, for $1 \le i \le l$,

$$\begin{aligned} n_i f_k &= f_k, \qquad \text{if } 1 \le k \le l,\ k \ne i \\ n_i f_{-k} &= f_{-k}, \qquad \text{if } 1 \le k \le l,\ k \ne i \\ n_i f_i &= f_{-i}, \\ n_i f_{-i} &= -f_i \end{aligned} \tag{5.26}$$

Similarly, for $1 \le i \le l$, define a monomial matrix $m_i \in GL(2l, K)$ as follows:

$$\begin{aligned} m_i f_k &= f_k, \qquad \text{if } 1 \le k \le l,\ k \ne i \\ m_i f_{-k} &= f_{-k}, \qquad \text{if } 1 \le k \le l,\ k \ne i \\ m_i f_i &= -f_{-i}, \\ m_i f_{-1} &= f_i \end{aligned}$$

Then it is easy to see that $n_i^{-1} = {}^t n_i = m_i$.

Now, we wish to show that $n_i, 1 \le i \le l$ lies in G, so consider how ${}^t n_i s n_i = m_i s n_i$ acts on the basis $\{f_{\pm j}\}_{1 \le j \le l}$. If $1 \le k \le l$ with $k \ne i$, then

$$(m_i s n_i) f_k = (m_i s) f_k = -m_i f_{-k} = -f_{-k}$$

which is clearly equal to $s f_k$. Similarly,

$$(m_i s n_i) f_{-k} = (m_i s) f_{-k} = m_i f_k = f_k$$

which is equal to sf_{-k}. Furthermore, for the case $k = i$,

$$(m_i s n_i) f_i = (m_i s) f_{-i} = m_i f_i = -f_{-i}$$

which is equal to sf_i, and

$$(m_i s n_i) f_{-i} = -(m_i s) f_i = m_i f_{-i} = f_i$$

which is equal to sf_{-i}. Therefore, ${}^t n_i s n_i = s$, and so $n_i \in G$ for all $1 \leq i \leq l$. Indeed, we can say more: since n_i is monomial, the results from §5.1.5 tell us that $n_i t n_i^{-1} \in D(2l, K)$ for all $t \in D(2l, K)$. But if $t \in G$, then $n_i t n_i^{-1} \in G$, and so $n_i t n_i^{-1} \in D(2l, K) \cap G = T$, which means $n_i \in N_G(T)$.

We also wish to show that, given $1 \leq i, j \leq l$, we have $n_i n_j = n_j n_i$. Certainly this is true if $i = j$, so suppose $i \neq j$. Then, for $1 \leq k \leq l$, with $k \neq i, j$, we have

$$\begin{aligned}
&(m_i n_j n_i) f_k = f_k, \\
&(m_i n_j n_i) f_{-k} = f_{-k}, \\
&(m_i n_j n_i) f_i = m_i n_j f_{-i} = m_i f_{-i} = f_i, \\
&(m_i n_j n_i) f_{-i} = -m_i n_j f_i = -m_i f_{-i} = f_{-i}, \\
&(m_i n_j n_i) f_j = -m_i n_j f_j = m_i f_{-j} = f_{-j}, \\
&(m_i n_j n_i) f_{-j} = -m_i n_j f_{-j} = -m_i f_j = -f_j
\end{aligned}$$

and so $m_i n_j n_i = n_j$, whence $n_j n_i = n_i n_j$. We also note here that n_i^2 is the matrix whose i^{th} column is $-f_i$, whose $-i^{\text{th}}$ column is $-f_{-i}$, and whose k^{th} column is f_k where $1 \leq \pm k \leq l$ and $k \neq i, -i$.

Suppose $F \subset \{1, \ldots, l\}$. Then define a matrix

$$n_F = \prod_{i \in F} n_i$$

This is well defined since $n_i n_j = n_j n_i$. Moreover, since it is the product of elements in $N_G(T)$, n_F itself lies in $N_G(T)$. It is not difficult to see that, if $F \neq F'$ are both subsets of $\{1, \ldots, l\}$, then $n_F \neq n_{F'}$. Indeed, if $k \in F - F'$, then

$$n_F f_k = f_{-k}$$

but

$$n_{F'} f_k = f_k$$

Finally, we note that $n_F n_{F'} = n_{F'} n_F$, and ensure that we set $n_\emptyset$ to be the identity.

The next step is to define a subset of W as follows:

$$Y = \{ n_F T \mid F \subset \{1, \ldots, l\} \}$$

Suppose $F, F' \subset \{1, \ldots, l\}$. Then it is not in general the case that that $n_F n_{F'} = n_{F \cup F'}$, since if $i \in F \cap F'$, then there is a diagonal term n_i^2 in the product, which does not arise in the matrix $n_{F\cup F'}$. However, passing to the level of cosets, by the above arguments we see that

$$(n_F T)(n_{F'} T) = n_{(F\cup F' - F\cap F')} T$$

Therefore Y is closed under multiplication. Moreover, from this we see that $(n_F T)(n_F T) = n_\emptyset T = T$, and so Y is closed under taking of inverses, and is therefore a group. Since $n_F n_{F'} = n_{F'} n_F$, then Y is abelian. We should also note that if $F \neq F'$, then $n_F T \neq n_{F'} T$

We can say more about this group $Y \subset W$; if $p_\sigma T$ is an arbitrary element of W, then it is a simple calculation to show

$$p_\sigma n_F p_{\sigma^{-1}} T = n_{\sigma(F)} T \tag{5.27}$$

and therefore Y is normal in W. Indeed, if $x = p_\sigma t \in N_G(T)$ for some $t \in T$, then equation (5.25) tells us that $\sigma\gamma = \gamma\sigma$, where $\gamma \in \mathfrak{S}_{2l}$ is the permutation defined in (5.23). Note that, in fact, γ is precisely the permutation which, in our index notation, sends i to $-i$ for $-l \leq i \leq l$. It follows that if $x = p_\sigma t \in N_G(T)$, then

$$\sigma(-i) = \sigma\gamma(i) = \gamma\sigma(i) = -\sigma(i) \tag{5.28}$$

We return to equation (5.27). Since $(n_{\sigma(F)} T)^{-1} = n_{\sigma(F)} T$, it suffices to prove that

$$p_\sigma n_F p_{\sigma^{-1}} n_{\sigma(F)} T = T$$

which is to say, we need to show that the matrix $p_\sigma n_F p_{\sigma^{-1}} n_{\sigma(F)}$ is diagonal. To do so, we will consider the action of this matrix on the column vectors f_i for $-l \leq i \leq l$. In the first place, suppose $i \notin \sigma(F)$, then $\sigma^{-1}(i) \notin F$, and so

$$p_\sigma n_F p_{\sigma^{-1}} n_{\sigma(F)}(f_i) = p_\sigma n_F p_{\sigma^{-1}}(f_i) = p_\sigma n_F (f_{\sigma^{-1}(i)}) = p_\sigma (f_{\sigma^{-1}(i)}) = f_i$$

On the other hand, if $i \in \sigma(F)$, then $\sigma^{-1}(i) \in F$, and so

$$\begin{aligned} p_\sigma n_F p_{\sigma^{-1}} n_{\sigma(F)}(f_i) &= p_\sigma n_F p_{\sigma^{-1}}(f_{-i}) = p_\sigma n_F (f_{\sigma^{-1}(-i)}) \\ &= p_\sigma n_F (f_{-\sigma^{-1}(i)}) = -p_\sigma (f_{\sigma^{-1}(i)}) \\ &= -f_i \end{aligned}$$

where we used the fact given in equation (5.28) that $\sigma(-i) = -\sigma(i)$. So we have shown that $p_\sigma n_F p_{\sigma^{-1}} n_{\sigma(F)}$ is diagonal, and so have verified equation (5.27). The conclusion, then, is that Y is normal in W.

We construct a map

$$\begin{aligned} q : (\mathbb{Z}/2\mathbb{Z})^l &\longrightarrow Y \\ (\epsilon_1, \ldots, \epsilon_l) &\longmapsto n_F T \end{aligned}$$

where $i \in F$ if $\epsilon_i = 1$ and $i \notin F$ if $\epsilon_i = 0$. Now, this map is certainly surjective, and so is bijective, by comparing cardinalities of the sets involved. Suppose $\epsilon_i + \epsilon_i' = 0$. Then either $\epsilon_i = \epsilon_i' = 0$, in which case $i \notin F \cup F'$, or $\epsilon_i = \epsilon_i' = 1$, in which case $i \in F \cap F'$. Therefore, if $\epsilon_i + \epsilon'_i = 0$, then $i \notin F \cup F' - F \cap F'$. On the other hand, if $\epsilon_i + \epsilon'_i = 1$, then either $\epsilon_i = 1$ and $\epsilon'_i = 0$, in which case $i \in F - F'$, or $\epsilon_i = 0$ and $\epsilon'_i = 1$, in which case $i \in F' - F$. We have shown, then, that

$$q((\epsilon_1, \dots, \epsilon_l)(\epsilon'_1, \dots, \epsilon'_l)) = n_{F \cup F' - F \cap F'} T = n_F T n_{F'} T q(\epsilon_1, \dots, \epsilon_l) q(\epsilon'_1, \dots, \epsilon'_l)$$

and so q is a bijective group homomorphism, and therefore $Y \cong (\mathbb{Z}/2\mathbb{Z})^l$.

Finally, we claim that $W = Y \rtimes w(\mathfrak{S}_l)$. Note firstly that, if $x \in Y \cap w(\mathfrak{S}_l)$, say $x = n_F T = p(\sigma)T$, then $p(\sigma^{-1})n_F \in T$. In particular, for each $1 \leq i \leq l$, we would have $(p(\sigma^{-1})n_F)f_i = \lambda_i f_i$ for some $\lambda_i \in K^*$. We know $n_F f_i = f_{-i}$ if $i \in F$ by (5.26), in which case

$$(p(\sigma^{-1})n_F)f_i = p(\sigma^{-1})f_{-i} = f_{-\sigma(i)} \neq \lambda_i f_i$$

Therefore, if $p(\sigma^{-1})n_F \in T$, then $i \notin F$ for all $1 \leq i \leq l$, which means $n_F = 1$, and so $p(\sigma^{-1}) \in T$. But we already saw that this is only possible if $\sigma^{-1} = e$, and so we conclude that $Y \cap w(\mathfrak{S}_l) = \{e\}$.

It remains to show that $W = Y \cdot w(\mathfrak{S}_l)$, so let $xT \in W$ with $x \in N_G(T)$. We saw above that each $x \in N_G(T)$ is monomial, and so there is an element $\sigma \in \mathfrak{S}_{2l}$ such that $p_{\sigma^{-1}}x = t$ is diagonal. We also recall from (5.25) that σ is such that it commutes with the permutation γ, defined in (5.23). We now define a set $F \subset \{1, \dots, l\}$ as follows:

$$F = \{i \mid \sigma(i) > l\}$$

Now, for each element $i \in F$, define a permutation $\tau_i \in \mathfrak{S}_{2l}$ to be the transposition

$$\tau_i = (\sigma(i), 2l + 1 - \sigma(i))$$

It is certainly true that $\tau_i \tau_j = \tau_j \tau_i$ for $i, j \in F$, since if $i \neq j$, then $\sigma(i) \neq \sigma(j)$. Therefore, we can easily define another element of $\mathfrak{S}_{2l}$

$$\mu = \prod_{i \in F} \tau_i$$

irrespective of the order of multiplication. Suppose now that $1 \leq j \leq l$. If $j \in F$, then $\sigma(j) > l$ and

$$\mu\sigma(j) = \tau_j \sigma(j) = 2l + 1 - \sigma(j) \leq l$$

On the other hand, let $j \notin F$. Suppose there exists an element $i \in \{1, \dots, l\}$ such that $\sigma(j) = 2l + 1 - \sigma(i)$. This implies

$$\sigma(j) = \gamma\sigma(i) = \sigma\gamma(i)$$

where the second equality is by equation (5.25). We conclude that $j = \gamma(i)$, but this is impossible since we are assuming both i and j lie in $\{1, \dots, l\}$. It follows, therefore, that if $j \notin F$, then

$$\mu\sigma(j) = \sigma(j) \in \{1, \dots, l\}$$

Putting these observations together we see that $\mu\sigma$ stabilises the set $\{1, \ldots, l\}$, and so we can define a permutation $\nu \in \mathfrak{S}_l$ such that $\mu\sigma(j) = \nu(j)$ for all $j \in \{1, \ldots, l\}$. In particular,

$$\nu(i) = \begin{cases} 2l + 1 - \sigma(i) & \text{if } i \in F \\ \sigma(i) & \text{if } i \notin F \end{cases}$$

We now claim that

$$xT = n_{\sigma(F)}p(\nu)T \tag{5.29}$$

But $xT = p_\sigma tT$, for some diagonal t, and so it suffices to show that the matrix $p_{\sigma^{-1}}n_{\sigma(F)}p(\mu)$ is diagonal. We do so by checking the way it acts on the basis elements f_i.

Indeed, suppose $i \in \{1, \ldots, l\}$. It is easy to check that $p(\nu)f_i = f_{\nu(i)}$. Therefore, if $i \in F$, we have $\sigma(i) \in \sigma(F)$, and so

$$\begin{aligned} p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_i &= p_{\sigma^{-1}}n_{\sigma(F)}f_{\nu(i)} = p_{\sigma^{-1}}n_{\sigma(F)}f_{2l+1-\sigma(i)} \\ &= -p_{\sigma^{-1}}f_{\sigma(i)} = -f_i \end{aligned}$$

On the other hand, if $i \notin F$, we have $\sigma(i) \notin \sigma(F)$ and $\mu\sigma(i) = \sigma(i)$, so

$$\begin{aligned} p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_i &= p_{\sigma^{-1}}n_{\sigma(F)}f_{\nu(i)} = p_{\sigma^{-1}}n_{\sigma(F)}f_{\sigma(i)} \\ &= p_{\sigma^{-1}}f_{\sigma(i)} = f_i \end{aligned}$$

Next we consider the action on f_{-i} for $i \in \{1, \ldots, l\}$. To do so we first need to calculate $p(\nu)f_{-i}$. Indeed

$$\begin{pmatrix} p_\nu & 0 \\ 0 & p_\pi p_\nu p_\pi \end{pmatrix} \begin{pmatrix} 0 \\ e_{l+1-i} \end{pmatrix} = \begin{pmatrix} 0 \\ p_\pi p_\nu p_\pi e_{l+1-i} \end{pmatrix}$$

where e_j is the l-dimensional column vector with 1 in the j^{th} entry and zeros elsewhere. Moreover, if $i \notin F$, then

$$p_\pi p_\nu p_\pi e_{l+1-i} = p_\pi p_\nu e_i = p_\pi e_{\nu(i)} = p_\pi e_{\sigma(i)} = e_{l+1-\sigma(i)} \tag{5.30}$$

since π sends j to $l + 1 - j$. It follows that $p(\nu)f_{-i} = f_{2l+1-\sigma(i)}$. On the other hand, if $i \in F$, then

$$p_\pi p_\nu p_\pi e_{l+1-i} = p_\pi p_\nu e_i = p_\pi e_{\nu(i)} = p_\pi e_{2l+1-\sigma(i)} = e_{\sigma(i)-l} \tag{5.31}$$

and so it follows that $p(\nu)f_{-i} = f_{\sigma(i)}$.

Returning to the verification of (5.29), we consider the product $p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_{-i}$ for $i \in \{1, \ldots, l\}$. If $i \notin F$, then

$$\begin{aligned} p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_{-i} &= p_{\sigma^{-1}}n_{\sigma(F)}f_{2l+1-\sigma(i)} \\ &= p_{\sigma^{-1}}f_{-\sigma(i)} = f_{-i} \end{aligned}$$

since $\sigma^{-1}(-j) = -\sigma^{-1}(j)$ by (5.28). On the other hand, if $i \in F$, then

$$\begin{aligned} p_{\sigma^{-1}} n_{\sigma(F)} p(\nu) f_{-i} &= p_{\sigma^{-1}} n_{\sigma(F)} f_{\sigma(i)} \\ &= -p_{\sigma^{-1}} f_{-\sigma(i)} = -f_{-i} \end{aligned}$$

We have shown that $p_{\sigma^{-1}} n_{\sigma(F)} p(\nu) = t'$ is diagonal, and thus

$$n_{\sigma(F)} p(\nu) = p_\sigma t' = p_\sigma t(t^{-1}t') = x(t^{-1}t') \tag{5.32}$$

But

$$t^{-1}t' = (x^{-1}p_\sigma)(p_{\sigma^{-1}} n_{\sigma(F)} p(\nu)) = x^{-1} n_{\sigma(F)} p(\nu)$$

and since all of these latter terms lie in G, so does $t^{-1}t'$. Therefore, in W, equaiton (5.32) becomes

$$n_{\sigma(F)} p(\nu) T = x(t^{-1}t')T = xT$$

and so $W \subset Y \cdot w(\mathfrak{S}_l)$. We can therefore conclude that

$$W = Y \rtimes w(\mathfrak{S}_l) \cong (\mathbb{Z}/2\mathbb{Z})^l \rtimes \mathfrak{S}_l$$

5.3.7 Dynkin Diagram

To calculate the Cartan integers for the root system, we again need to calculate explicitly the reflections $\sigma_{\alpha_i} = \sigma_i$ for each $\alpha_i \in \Delta$, as defined as defined in §5.3.3. Recall also that $Sp(2, K) = SL(2, K)$, which deals with the case $l = 1$, so we assume that $l \geq 2$.

Suppose firstly that $1 \leq i \leq l - 1$. That is,

$$\alpha_i : \operatorname{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1})$$

The appropriate candidate for the reflection σ_i corresponds to $(i, i+1)$ under the inclusion $W \subset \mathfrak{S}_{2l}$. That is, σ_i acts on elements of the torus $t \in T$ by swapping the i^{th} and $(i+1)^{\text{th}}$ diagonal entries.

We first consider the case $i < l - 1$. We apply the permutation $(i, i+1)$ to the diagonal elements t, and with the help of the formulae given in §5.3.3, it is easy to calculate the following:

$$\sigma_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } j = i \\ \alpha_{i-1} + \alpha_i & \text{if } j = i - 1 \\ \alpha_i + \alpha_{i+1} & \text{if } j = i + 1 \\ \alpha_j & \text{otherwise} \end{cases}$$

Notice that this σ_i inverts α_i and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_i\}$. Therefore, by Lemma 3.5.2.4, this σ_i is indeed the reflection corresponding to α_i. Therefore, plugging this into equation (5.11),

$$\langle \alpha_j, \alpha_i \rangle \alpha_i = \alpha_j - \sigma_i(\alpha_j) = \begin{cases} 2\alpha_i & \text{if } j = i \\ -\alpha_i & \text{if } j = i - 1 \\ -\alpha_i & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

We move on to the computations for σ_{l-1}:

$$\sigma_{l-1}(\alpha_j) = \begin{cases} -\alpha_{l-1} & \text{if } j = l-1 \\ 2\alpha_{l-1} + \alpha_l & \text{if } j = l \\ \alpha_j & \text{otherwise} \end{cases}$$

Again, this σ_{l-1} inverts α_{l-1} and sends all other simple roots to the set $\Phi^+ - \{\alpha_{l-1}\}$, and so Lemma 3.5.2.4, says that this is indeed the reflection corresponding to α_{l-1}. Moreover,

$$\langle \alpha_j, \alpha_{l-1} \rangle \alpha_{l-1} = \alpha_j - \sigma_{l-1}(\alpha_j) = \begin{cases} 2\alpha_{l-1} & \text{if } j = l-1 \\ -2\alpha_{l-1} & \text{if } j = l \\ 0 & \text{otherwise} \end{cases}$$

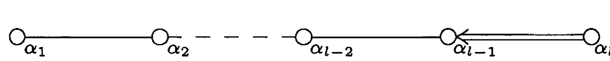
We now consider the reflection σ_l. Since $\alpha_l(t) = t_l^2$, the reflection which inverts this root but which stabilises $\Phi^+ - \{\alpha_l\}$ is evidently the transposition $(l, l+1) \in \mathfrak{S}_{2l}$. That is, the reflection acts on elements of the torus by inverting each of the l^{th} and the $(l+1)^{\text{th}}$ diagonal entries. Using the formulae in §5.3.3 again, we calculate

$$\sigma_l(\alpha_j) = \begin{cases} -\alpha_l & \text{if } j = l \\ \alpha_{l-1} + \alpha_l & \text{if } j = l-1 \\ \alpha_j & \text{otherwise} \end{cases}$$

Indeed, σ_l as defined here inverts α_l and sends the remaining simple roots to the set $\Phi^+ - \{\alpha_l\}$, and thus Lemma 3.5.2.4, confirms that this is indeed the reflection corresponding to α_l. Moreover,

$$\langle \alpha_j, \alpha_l \rangle \alpha_l = \alpha_j - \sigma_l(\alpha_j) = \begin{cases} 2\alpha_l & \text{if } j = l \\ -\alpha_l & \text{if } j = l-1 \\ 0 & \text{otherwise} \end{cases}$$

We are now ready to construct the Dynkin diagram for $G = Sp(2l, K)$, by computing the Cartan integers from the above formulae. We specifically note that $\langle \alpha_l, \alpha_{l-1} \rangle \big(\langle \alpha_{l-1}, \alpha_l \rangle\big)^{-1} = 2$, and so there is an arrow pointing from the node α_l to the node α_{l-1}, and a double bond joining them. The diagram is:



5.4 The Even Special Orthogonal Group

It is convenient to assume char$K \neq 2$. The even special orthogonal group $SO(2l, K)$ is defined to consist of all $x \in SL(2l, K)$ which satisfy

$$^t x s x = s$$

where
$$s = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$
and J is as above. Again, this condition defines a closed subgroup of $GL(2l, K)$, and moreover we saw in Proposition 1.3.2.8 that G is connected.

We can work through the standard linear algebraic calculations to show that $\mathscr{L}(G) = \mathfrak{g}$ consists of matrices of the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -J^t aJ \end{pmatrix} \,\middle|\, a, b, c \in \mathrm{Mat}(l, K),\ {}^t b = -JbJ, {}^t c = -JcJ \right\} \tag{5.33}$$

That is, b, c are skew-symmetric about the skew diagonal. See, for example, §8.13.4 of [3]. From (5.33) we also see that $\dim G = \dim(\mathfrak{g}) = l^2 + l(l-1) = 2l^2 - l$.

5.4.1 Maximal Torus

Let $T \subset GL(2l, K)$ be the set of matrices of the form

$$g = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_l & & & \\ & & & t_l^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \tag{5.34}$$

We wish to show that $T = D(2l, K) \cap SO(2l, K)$. Given an arbitrary diagonal matrix
$$g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$
with $x, y \in D(l, K)$, then g lies in $SO(2l, K)$ if and only if $gsg = s$, that is,

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & xJy \\ yJx & 0 \end{pmatrix} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$

So g is in $SO(2l, K)$ if and only if $xJy = J$. In particular,

$$x_{ii} J_{ij} y_{jj} = J_{ij}, \quad 1 \le i, j \le l$$

which, in turn, is equivalent to

$$x_{ii} y_{jj} = 1, \quad \forall i, j \text{ such that } i + j = l + 1 \tag{5.35}$$

So now suppose $g \in T$. Then clearly $g \in D(2l, K)$. Moreover, since g clearly satisfies (5.35), we have that $g \in SO(2l, K)$, which gives us $T \subset D(2l, K) \cap SO(2l, K)$. On the other hand, if $g \in D(2l, K) \cap SO(2l, K)$, then g is diagonal, satisfying (5.35). This tells us that $y_{11} = x_{nn}^{-1}, y_{22} = x_{n-1,n-1}^{-1}, \ldots, y_{nn} = x_{11}^{-1}$, which means that $g \in T$, as required.

Now, T is certainly a torus, since it is isomorphic to $D(l, K)$. Moreover, it is a torus in $GL(2l, K,)$, and is contained in the maximal torus of $D(2l, K)$ of $GL(2l, K) = G'$. We can argue in an identical fashion to §5.3.1 to show that $C_{G'}(T) = D(2l, K)$. Moreover, $C_G(T) = G \cap C_{G'}(T) = G \cap D(2l, K)$.

We wish to show that T is in fact a maximal torus of G. Suppose, then, that T' is a maximal torus of G such that $T \subset T'$. Certainly $T' \subset C_G(T)$, since T' is commutative and contains T, and so

$$T \subset T' \subset C_G(T) = G \cap D(2l, K) = T$$

from which we can conclude $T = T'$, and so T is maximal.

5.4.2 Root System

We continue with the notation $G = SO(2l, K)$, and, as above, $T = G \cap D(2l, K)$. Then certainly

$$\mathscr{L}(T) \subset \mathfrak{g} \cap \mathfrak{d}(2l, K) = \left\{ \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_l & & & \\ & & & -a_l & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix} \right\}$$

where $a_i \in K$ for $1 \leq i \leq l$. This last equality is simply by directly computing the intersection based on (5.33). But $\dim(\mathscr{L}(T)) = \dim T = l$, and this is clearly equal to the dimension of $\mathfrak{g} \cap \mathfrak{d}(2l, K)$, and so the above inclusion is equality. Moreover, Proposition 2.3.1.9 says that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathscr{L}(C_G(T))$, but this latter algebra is equal to $\mathscr{L}(T)$ since $T = C_G(T)$. Altogether, then, we have

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathscr{L}(T) = \mathfrak{g} \cap \mathfrak{d}(2l, K) \tag{5.36}$$

That is, the centraliser of the T-action on $\mathfrak{g}$ is equal to $\mathfrak{g} \cap \mathfrak{d}(2l, K)$.

Once more we label the standard basis of K^{2l} by $f_{\pm 1}, \ldots, f_{\pm l}$, such that, for $1 \leq i \leq l$, $f_i = e_i$ and $f_{-i} = e_{2l+1-i}$. We also persist with the notation $F_{i,j}$ for the matrix which has zero entries everywhere except the $(i, j)^{\text{th}}$ entry, which is 1, and where $-l \leq i, j \leq l$.

Now suppose $1 \leq i, j \leq l$, with $i \neq j$, and define the matrix $X_{\epsilon_i - \epsilon_j} = F_{i,j} - F_{-j,-i}$. That is,

$$X_{\epsilon_i - \epsilon_j} = \left(\begin{array}{c|c} E_{i,j} & 0 \\ \hline 0 & -E_{l+1-j,l+1-i} \end{array} \right)$$

where here the $E_{r,s}$ matrices are the $l \times l$ elementary matrices. Since $E_{l+1-j,l+1-i} = J^t E_{i,j} J$, comparison with (5.33) shows $X_{\epsilon_i - \epsilon_j} \in \mathfrak{g}$. Now, for any matrix

$t = \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots t_1^{-1}) \in T$,

$$tX_{\epsilon_i - \epsilon_j} t^{-1} = \frac{t_i}{t_j} X_{\epsilon_i - \epsilon_j}$$

so $X_{\epsilon_i - \epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i/t_j$$

Now define $X_{\epsilon_i + \epsilon_j} = F_{i,-j} - F_{j,-i}$ for $1 \leq i, j \leq l$, with $i \neq j$. That is,

$$X_{\epsilon_i + \epsilon_j} = \left(\begin{array}{c|c} 0 & E_{i,l+1-j} - E_{j,l+1-i} \\ \hline 0 & 0 \end{array} \right)$$

The upper right submatrix is evidently skew symmetric about its skew diagonal, and so (5.33) tells us that $X_{\epsilon_i + \epsilon_j} \in \mathfrak{g}$. Moreover, for any matrix $t = \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots t_1^{-1}) \in T$,

$$tX_{\epsilon_i + \epsilon_j} t^{-1} = t_i t_j X_{\epsilon_i + \epsilon_j}$$

so $X_{\epsilon_i + \epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j$$

Similarly, define $X_{-\epsilon_i - \epsilon_j} = F_{-j,i} - F_{-i,j}$ for $1 \leq i, j \leq l$, with $i \neq j$. That is,

$$X_{-\epsilon_i - \epsilon_j} = \left(\begin{array}{c|c} 0 & 0 \\ \hline E_{l+1-j,i} - E_{l+1-i,j} & 0 \end{array} \right)$$

The lower submatrix is skew symmetric about its skew diagonal, and so (5.33) tells us that $X_{-\epsilon_i - \epsilon_j} \in \mathfrak{g}$. Furthermore, for any matrix $t = \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots t_1^{-1}) \in T$,

$$tX_{-\epsilon_i - \epsilon_j} t^{-1} = t_i^{-1} t_j^{-1} X_{-\epsilon_i - \epsilon_j}$$

so $X_{-\epsilon_i - \epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto \frac{1}{t_i t_j}$$

We have therefore shown that the following $2l^2 - 2l$ distinct characters are roots of $G = Sp(2l, K)$:

$$\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i, j \leq l, \, i \neq j \tag{5.37}$$

$$\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j, \quad 1 \leq i, j \leq l, \, i \neq j \tag{5.38}$$

$$\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}, \quad 1 \leq i, j \leq l, \, i \neq j \tag{5.39}$$

We can now appeal to Lemma 3.2.3.2 again, which says $\mathrm{Card}(\Phi) \leq \dim(G) - \dim(C_G(T)) = 2l^2 - l - l$, and so the above list of roots exhausts the elements of Φ. In turn we can conclude that $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$.

By comparing the spaces $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ computed above, and (5.33), we see that we have produced a decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{5.40}$$

5.4.3 Simple Roots

If $l = 1$, then an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of G is such that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2ac & ad+bc \\ ad+bc & 2bd \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so we have equations $ac = bd = 0$ and $ad + bc = 1$. But since $G \subset SL(2, K)$ by definition, we also have $ad - bc = 1$, which forces $ad = 1$ and $b = c = 0$. Therefore, elements of $SO(2, K)$ are of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

and so $SO(2, K) \cong GL(1, K)$, and so we have already dealt with this case in §5.1.

If $l = 2$, then $SO(2l, K)$ has precisely four roots, namely

$$\begin{aligned} \alpha_1 &: \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \longmapsto t_1/t_2 \\ -\alpha_1 &: \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \longmapsto t_2/t_1 \\ \alpha_2 &: \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \longmapsto t_1 t_2 \\ -\alpha_2 &: \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \longmapsto (t_1 t_2)^{-1} \end{aligned}$$

It is immediate that the reflection σ_{α_1} associated with α_1 acts on elements of the torus by switching the first and second diagonal entries, and so $\sigma_{\alpha_1}(\alpha_2) = \alpha_2$. Similarly, the reflection σ_{α_2} associated with α_2 acts on elements of the torus by swapping the first diagonal entry and the third, as well as the second diagonal entry and the fourth. It follows that

$$\sigma_{\alpha_2}(\alpha_1)(\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1})) = \alpha_1(\mathrm{diag}(t_2^{-1}, t_1^{-1}, t_1, t_2^{-1})) = t_2^{-1}/t_1^{-1} = t_1/t_2$$

and so $\sigma_{\alpha_2}(\alpha_1) = \alpha_1$. We can therefore partition Φ into two distinct sets $\Phi_1 = \{\alpha_1, -\alpha_1\}$ and $\Phi_2 = \{\alpha_2, -\alpha_2\}$, such that all reflections associated to roots in one of these sets act trivially on the roots of the other set. Such subsets are called *orthogonal*, and the root system Φ is called *reducible*. Each of these subsystems has a base, say $\Delta_1 = \{\alpha_1\}$ and $\Delta_2 = \{\alpha_2\}$. Note further that Φ_1 and Φ_2 are root systems of the same form as that of $SL(2, K)$.

If $l \geq 3$, we will now show that we get irreducible root systems. We begin by defining a root

$$\alpha_i : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots t_1^{-1}) \longmapsto t_i/t_{i+1}$$

for $1 \leq i \leq l - 1$, and define

$$\alpha_l : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots t_1^{-1}) \longmapsto t_{l-1} t_l$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. We wish to show that Δ is a base for Φ, and so we need to show that each $\alpha \in \Phi$ can be expressed in the form $\alpha = \sum c_i \alpha_i$ where the c_i's are scalars with the same sign. Let $\alpha \in \Phi$ be a root of the form given above in (5.37), say

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i/t_j$$

Then, if $i < j$,

$$\alpha = \alpha_i + \cdots + \alpha_{j-1}$$

while on the other hand, if $j < i$, then

$$\alpha = -(\alpha_j + \cdots + \alpha_{i-1})$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.38), say

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j$$

If $1 \leq i \leq l-2, j = l-1$, then

$$\alpha = \alpha_i + \cdots + \alpha_{l-1} + \alpha_l$$

If $1 \leq i \leq l-2, j = l$, then

$$\alpha = \alpha_i + \cdots + \alpha_{l-2} + \alpha_l$$

If $1 \leq i < j \leq l-2$, then

$$\alpha = (\alpha_i + \cdots + \alpha_{l-2}) + (\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.39), say

$$\alpha : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}$$

If $1 \leq i \leq l-2, j = l-1$, then

$$\alpha = -(\alpha_i + \cdots + \alpha_{l-1} + \alpha_l)$$

If $1 \leq i \leq l-2, j = l$, then

$$\alpha = -(\alpha_i + \cdots + \alpha_{l-2} + \alpha_l)$$

If $1 \leq i < j \leq l-2$, then

$$\alpha = -(\alpha_i + \cdots + \alpha_{l-2}) - (\alpha_j + \cdots + \alpha_{l-1}) - \alpha_l$$

It follows that Δ is a base for Φ. The associated set of positive roots consists of the following $l(l-1)$ distinct roots:

$$\begin{aligned}
&\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i < j \leq l, \\
&\mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j, \quad 1 \leq i, j \leq l,\ i \neq j
\end{aligned}$$

5.4.4 Radical and Unipotent Radical

In this section, we will repeat the argument given in §5.3.4, so we must start by explicitly calculating some Lie brackets in $\mathfrak{g}$. In particular, let $\alpha \in \Phi$, and take a nonzero element $\mathsf{x} \in \mathfrak{g}_\alpha$. Our computation of Φ in §5.4.2 above shows us that $\Phi = -\Phi$, and so $-\alpha$ is also a root of G. Take a nonzero element $\mathsf{y} \in \mathfrak{g}_{-\alpha}$. We wish to show that the bracket $[\mathsf{x}, \mathsf{y}]$ is nonzero for each $\alpha \in \Phi$.

Suppose firstly that α is of the form given in (5.37). Then, since $\dim \mathfrak{g}_\alpha = 1$, in the notation of §5.4.2, x is a nonzero multiple of $X_{\epsilon_i - \epsilon_j} = F_{i,j} - F_{-j,-i}$, with $i \neq j$, and similarly y is a nonzero multiple of $X_{\epsilon_j - \epsilon_i}$. By direct calculation, we have

$$[X_{\epsilon_i - \epsilon_j}, X_{\epsilon_j - \epsilon_i}] = F_{i,i} - F_{j,j} + F_{-j,-j} - F_{-i,-i}$$

which is not zero, since $i \neq j$. Since $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of $[X_{\epsilon_i - \epsilon_j}, X_{\epsilon_j - \epsilon_i}]$, it is also nonzero.

Suppose now that α is of the form given in (5.38). Then the fact that $\dim \mathfrak{g}_\alpha = 1$ implies that x is a nonzero multiple of $X_{\epsilon_i + \epsilon_j} = F_{i,-j} - F_{j,-i}$, and likewise y is a nonzero multiple of $X_{-\epsilon_i, -\epsilon_j} = F_{-j,i} - F_{-i,j}$. By direct calculation,

$$[X_{\epsilon_i + \epsilon_j}, X_{-\epsilon_j - \epsilon_i}] = F_{i,i} + F_{j,j} - F_{-j,-j} - F_{-i,-i}$$

which is nonzero for all i, j. Since $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of the above expression, it too is nonzero. Finally, we observe that the case where α is of the form given in (5.39) is a consequence of the above calculation, since a root of this form is inverse to one of the form (5.38). We have therefore shown that if $\mathsf{x} \in \mathfrak{g}_\alpha$ is not zero, and $\mathsf{y} \in \mathfrak{g}_{-\alpha}$ is not zero, then $[\mathsf{x}, \mathsf{y}] \neq 0$.

In order to compute $R_u(G) = U$, we employ the same strategy as in §5.3.4, which is to consider the ideal $\mathscr{L}(U) = \mathfrak{u}$ which consists of nilpotent elements. Consider an element $\mathsf{x} \in \mathfrak{u}$. Since $\mathsf{x} \in \mathfrak{g}$, we can use (5.40) to express this element as $\mathsf{x} = \mathsf{x}_0 + \sum \mathsf{x}_\alpha$ where $\mathsf{x}_0 \in \mathfrak{t}$ and $\mathsf{x}_\alpha \in \mathfrak{g}_\alpha$. Moreover, we see again that

$$t^i.\mathsf{x} = \mathsf{x}_0 + \sum \alpha^i(t)\mathsf{x}_\alpha \in \mathfrak{u}$$

for all i and all $t \in T$. Again we note given any elements $\alpha, \beta \in \Phi$, we can choose an element $t \in T$ such that $\alpha^i(t) \neq \beta^i(t)$ for all $i > 0$. We can then repeat the argument given in §5.3.4, to show that $\mathsf{x}_0 \in \mathfrak{u}$, and $\mathsf{x}_\alpha \in \mathfrak{u}$ for all $\alpha \in \Phi$.

We therefore have $\mathsf{x}_0 \in \mathfrak{u} \cap \mathfrak{t}$. But $\mathfrak{u}$ consists of nilpotent elements, and $\mathfrak{t}$ consists of semisimple elements, and so $\mathsf{x}_0 = 0$. On the other hand, we have $\mathsf{x}_\alpha \in \mathfrak{u} \cap \mathfrak{g}_\alpha$, so suppose this element is nonzero. Select a nonzero element $\mathsf{y} \in \mathfrak{g}_{-\alpha}$, and so Lemma 3.4.3.1 tells us that $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{g}_0 = \mathfrak{t}$. On the other hand, we observed above that $\mathfrak{u}$ is an ideal, and so $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{u}$ since $\mathsf{x}_\alpha \in \mathfrak{u}$. We have shown, then, that $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{u} \cap \mathfrak{t}$, and so, being both semisimple and nilpotent, we are forced to conclude that $[\mathsf{x}_\alpha, \mathsf{y}] = 0$. But this contradicts the argument which began this section, and so it follows that $\mathsf{x}_\alpha = 0$ for all $\alpha \in \Phi$.

We have therefore shown that $\mathfrak{u} = \{0\}$. In particular, $0 = \dim \mathfrak{u} = \dim U$. But U is connected, and so must be trivial, and thus G is reductive.

Again, we can say more. We refer once more to Theorem 3.4.3.2(6), which says $Z(G)^\circ = \left(\bigcap_{\alpha\in\Phi} T_\alpha\right)^\circ$. Suppose, then, that $t = \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \in T$ is such that $\alpha(t) = 1$ for all $\alpha \in \Phi$. In particular, $t_i/t_j = 1$ and $t_it_j = 1$ for all i, j. Therefore $t = \pm I$ where I is the identity matrix. It follows that the connected component of $Z(G)$ is trivial, but since G is reductive, we can apply Lemma 3.1.1.9 which tells us $R(G) = Z(G)^\circ$, and so we conclude $R(G)$ is trivial, and therefore G is semisimple.

5.4.5 Borel Subgroup

Let $B \subset T(2l, K)$ be the set of matrices of the form

$$g = \begin{pmatrix} x & xJz \\ 0 & J{}^tx^{-1}J \end{pmatrix}$$

where x is upper triangular, and z is skew-symmetric. We wish to show that $B = T(2l, K) \cap SO(2l, K)$.

Firstly, suppose we have an arbitrary upper triangular matrix

$$g = \begin{pmatrix} x & y \\ 0 & x' \end{pmatrix}$$

where $x, x' \in T(l, K)$. Then $g \in SO(2l, K)$ if and only if ${}^tgsg = s$, or, equivalently,

$$\begin{pmatrix} {}^tx & 0 \\ {}^ty & {}^tx' \end{pmatrix} \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x' \end{pmatrix} = \begin{pmatrix} 0 & {}^tJx' \\ {}^tx'Jx & {}^tx'Jy + {}^tyJx' \end{pmatrix} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$$

which holds if and only if

1. ${}^txJx' = J$, and
2. ${}^tx'Jy = -{}^tyJx'$

Now suppose $g \in B$. Then certainly g is triangular. Moreover,

$${}^txJx' = ({}^txJ)(J{}^tx^{-1}J) = J$$

and so g satisfies (1). Secondly,

$${}^tx'Jy = {}^t(J{}^tx^{-1}J)J(xJz) = (Jx^{-1}J)J(xJz) = z \tag{5.41}$$

while

$$-{}^tyJx' = -{}^t(xJz)J(J{}^tx^{-1}J) = -(-zJ{}^tx)J(J{}^tx^{-1}J) = z \tag{5.42}$$

and so, combining equations (5.41) and (5.42), we see that g satisfies condition (2) above. Therefore g lies in $SO(2l, K)$, which tells us $B \subset T(2l, K) \cap SO(2l, K)$.

On the other hand, suppose $g \in T(2l, K) \cap SO(2l, K)$. Then g satisfies conditions (1) and (2) above. Indeed, condition (1) tells us $x' = J^t x^{-1} J$, and, if we define a $l \times l$ matrix $z = {}^t x' J y$, then condition (2) tells us that z is skew-symmetric. Moreover,

$$z = {}^t x' J y = {}^t(J^t x^{-1} J)(Jy) = J x^{-1} y$$

and so $y = xJz$. We have shown that $g \in B$, and so $B = T(2l, K) \cap SO(2l, K)$.

Now, certainly B is solvable, being a subgroup of $T(2l, K)$. Moreover, B is isomorphic as a variety to $T(l, K) \times X_l$, where X_l^a here denotes the variety consisting of $l \times l$ antisymmetric, invertible matrices. Since X_l^a is isomorphic to $\mathbf{A}^{n(n-1)/2}$, it is an irreducible variety, and since $T(l, K)$ is connected it follows that B is connected, too. Additionally, this isomorphism gives

$$\dim B = \dim T(l, K) + \dim X_l^a = l(l+1)/2 + l(l-1)/2 = l^2$$

Being solvable and connected, B lies in a Borel subgroup, B' say. Since G is reductive, we can again apply Theorem 3.4.3.2 to give us

$$\dim B' = \dim G - \frac{1}{2}\mathrm{Card}\Phi = 2l^2 - l - \frac{1}{2}(2l^2 - 2l) = l^2$$

Therefore $\dim B = \dim B'$, and since $B \subset B'$, we conclude that $B = B'$, and so B is a Borel subgroup of G.

Now it is clear that, since $B \subset T(2l, K)$, we have $\mathscr{L}(B) \subset \mathfrak{t}(2l, K)$ and since $B \subset SO(2l, K)$, we have $\mathscr{L}(B) \subset \mathfrak{so}(2l, K)$, which means $\mathscr{L}(B) \subset \mathfrak{t}(2l, K) \cap \mathfrak{so}(2l, K)$. But by direct observation with (5.33), we see that

$$\mathfrak{t}(2l, K) \cap \mathfrak{so}(2l, K) = \left\{ \begin{pmatrix} a & b \\ 0 & -J^t a J \end{pmatrix} \,\middle|\, a \in \mathfrak{t}(l, K), {}^t b = -JbJ \right\}$$

which, in turn, gives

$$\dim \left(\mathfrak{t}(2l, K) \cap \mathfrak{so}(2l, K)\right) = \frac{1}{2}l(l+1) + \frac{1}{2}l(l-1) = l^2$$

It follows that $\mathscr{L}(B) = \mathfrak{t}(2l, K) \cap \mathfrak{sp}(2l, K)$. It is now obvious that the roots of B are precisely the positive roots described in (5.22), and so the base given in §5.4.3 is equal to $\Delta(B)$ for $B = T(2l, K) \cap G$.

5.4.6 Weyl Group

We first calculate the normaliser $N_G(T)$ for $T = G \cap D(2l, K)$. By a similar argument identical to that of §5.3.6, it turns out that $N_G(T) = G \cap N_{GL(2l,K)}\left(D(2l, K)\right)$. Therefore elements of $N_G(T)$ are monomial matrices which lie in G. To determine what such elements look like, we first note that the matrix $s = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$ is precisely equal to the permutation matrix p_γ, where, as above, γ is the permutation

$$\gamma = (1, 2l)(2l-1, 2) \cdots (i, 2l+1-i) \cdots$$

Therefore, if $x = p_\sigma t$ for some $\sigma \in \mathfrak{S}_{2l}$ and diagonal matrix t, then

$$ {}^t x s x = {}^t(p_\sigma t) p_\gamma (p_\sigma t) = t p_{\sigma^{-1}} p_\gamma p_\sigma t = t' p_\sigma p_\gamma p_\sigma \tag{5.43} $$

for some diagonal t', where this final equality again follows from the fact observed in §5.1.5 that permutation matrices normalise diagonal matrices. So if ${}^t x s x = s$, then it follows that

$$ \gamma = \sigma^{-1} \gamma \sigma \tag{5.44} $$

Thus, given any element $x \in N_G(T)$, we can express x in the form $p_\sigma t$ for some permutation σ which commutes with γ. Moreover, another look at equation (5.43) gives us

$$ {}^t x s x = t p_{\sigma^{-1}} p_\gamma p_\sigma t = t p_\gamma t $$

and so t also lies in G. But t is diagonal, too, and we saw in §5.4.1 that such elements are of the form given in (5.34), and in particular, $\det t = 1$. But $\det x = 1$, and so p_σ is an even permutation.

We now turn to the computation of $W = N_G(T)/T$. For each $\sigma \in \mathfrak{S}_l$, define the matrix $p(\sigma)$ in $GL(2l, K)$ as in §5.3.6:

$$ p(\sigma) = \begin{pmatrix} p_\sigma & 0 \\ 0 & p_\pi p_\sigma p_\pi \end{pmatrix} $$

Again, as in §5.3.6, $p(\sigma) \in N_G(T)$ for all $\sigma \in \mathfrak{S}_l$. Once more we define an injective map

$$ \begin{aligned} w : \mathfrak{S}_l &\longrightarrow W \\ \sigma &\longmapsto p(\sigma)T \end{aligned} $$

We wish to discover the remaining elements of W.

Using the basis $\{f_{\pm i}\}_{1 \le i \le l}$ for K^{2l} again, used above. W we define a monomial matrix $n_i \in GL(2l, K)$, for $1 \le i \le l$ as follows:

$$ \begin{aligned} n_i f_k &= f_k, && \text{if } 1 \le k \le l,\ k \ne i \\ n_i f_{-k} &= f_{-k}, && \text{if } 1 \le k \le l,\ k \ne i \\ n_i f_i &= f_{-i}, \\ n_i f_{-i} &= f_i \end{aligned} \tag{5.45} $$

It is easy to see that $n_i^{-1} = {}^t n_i = n_i$, and $n_i n_j = n_j n_i$ for all $i, j \in \{1, \dots, l\}$.

Using similar arguments to that in §5.3.6, it can be shown that ${}^t n_i s n_i = s$. However, it is not the case that $n_i \in G$, since $\det n_i = -1$, for all i. Nevertheless, if we go on to define a matrix

$$ n_F = \prod_{i \in F} n_i $$

for $F \subset \{1, \dots, l\}$, then it is clear $\mathrm{Card}(F)$ is even if and only if $\det n_F = 1$, in which case $n_F \in N_G(T)$.

The next step is to define a subset of W as follows:

$$Y = \{n_F T \,|\, F \subset \{1, \ldots, l\} \text{ with } \mathrm{Card}(F) \text{ even } \}$$

By imitating the relevant arguments from §5.3.6 again, we can deduce that Y is a group, and indeed is normal in W.

We construct a map

$$\begin{aligned} r : Y &\longrightarrow (\mathbb{Z}/2\mathbb{Z})^l \\ n_F T &\longmapsto (\epsilon_1, \ldots, \epsilon_l) \end{aligned}$$

where if $i \in F$ then $\epsilon_i = 1$ and if $i \notin F$ then $\epsilon_i = 0$.

Now, this map is certainly injective. Moreover, the construction here is inverse to that which we gave in in §5.3.6 and so it is easy to see that it is a homomorphism. However, r is not bijective here, but rather maps to precisely half the elements of $(\mathbb{Z}/2\mathbb{Z})^l$, namely those for which $\sum_{i=1}^{l} \epsilon_i$ is even. It follows that $Y \cong \mathrm{im}(r) = (\mathbb{Z}/2\mathbb{Z})^{l-1}$.

We now repeat the argument given at the end of §5.3.6, to show that $W = Y \rtimes w(\mathfrak{S}_l)$. We have already seen that Y is normal in W. Suppose now that $x \in Y \cap w(\mathfrak{S}_l)$, say $x = p(\sigma)T = n_F T$, so $p(\sigma^{-1})n_F \in T$. Then if $i \in F$, we have

$$(p(\sigma^{-1})n_F)f_i = p(\sigma^{-1})f_{-i} = f_{-\sigma(i)}$$

which contradicts the fact that $p(\sigma^{-1})n_F \in T$. Therefore n_F is the identity, and so $Y \cap w(\mathfrak{S}_l)$ is trivial.

It remains to show that $W = Y \cdot w(\mathfrak{S}_l)$, so let $xT \in W$ where $x = p_\sigma t \in N_G(T)$ for some diagonal matrix t and p_σ is the permutation matrix corresponding to $\sigma \in \mathfrak{S}_{2l}$. Recall that equation (5.44) tells us that σ commutes with γ, the permutation corresponding to $s = p_\gamma$. We also saw that a consequence of this relation is $\det x = \det t = \det p_\sigma = 1$. As in §5.3.6, we define a set $F \subset \{1, \ldots, l\}$ as follows:

$$F = \{i \,|\, \sigma(i) > l\}$$

and for each element $i \in F$, define a permutation $\tau_i \in \mathfrak{S}_{2l}$ to be the transposition

$$\tau_i = (\sigma(i), 2l + 1 - \sigma(i))$$

Arguing as in §5.3.6, we Define another element of $\mathfrak{S}_{2l}$

$$\mu = \prod_{i \in F} \tau_i$$

which is independent of the order of multiplication, and finally, the fact that $\gamma\sigma = \sigma\gamma$ again means that $\mu\sigma$ stabilises the set $\{1, \ldots, l\}$. We define the permutation $\nu \in \mathfrak{S}_l$ such that $\nu(i) = \mu\sigma(i)$ for all $j \in \{1, \ldots, l\}$. Again, we have

$$\nu(i) = \begin{cases} 2l + 1 - \sigma(i) & \text{if } i \in F \\ \sigma(i) & \text{if } i \notin F \end{cases}$$

and so equations (5.30) and (5.31) also hold in this setting.

We now claim that

$$xT = n_{\sigma(F)}p(\nu)T \tag{5.46}$$

Indeed, suppose $i \in \{1, \ldots, l\}$. If $i \notin F$, then

$$p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_i = p_{\sigma^{-1}}n_{\sigma(F)}f_{\nu(i)} = p_{\sigma^{-1}}n_{\sigma(F)}f_{\sigma(i)} = p_{\sigma^{-1}}f_{\sigma(i)} = f_i$$

and

$$p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_{-i} = p_{\sigma^{-1}}n_{\sigma(F)}f_{-\sigma(i)} = p_{\sigma^{-1}}f_{-\sigma(i)} = f_{\sigma^{-1}(\sigma(-i))} = f_{-i}$$

where the last equality is again a consequence of the fact that $\sigma\gamma = \gamma\sigma$. On the other hand, if $i \in F$,

$$\begin{aligned} p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_i &= p_{\sigma^{-1}}n_{\sigma(F)}f_{\nu(i)} = p_{\sigma^{-1}}n_{\sigma(F)}f_{2l+1-\sigma(i)} \\ &= p_{\sigma^{-1}}f_{\sigma(i)} = f_i \end{aligned}$$

and

$$\begin{aligned} p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)f_{-i} &= p_{\sigma^{-1}}n_{\sigma(F)}f_{2l+1-\sigma} = p_{\sigma^{-1}}f_{2l+1-\sigma(i)} \\ &= f_{\sigma^{-1}(-\sigma(i))} = f_{-i} \end{aligned}$$

We have therefore shown that $p_{\sigma^{-1}}n_{\sigma(F)}p(\nu)$ acts trivially on the standard basis elements, and so

$$n_{\sigma(F)}p(\nu) = p_\sigma$$

We know that $\det p(\nu) = \det p_\sigma = 1$, and so it follows that $\det n_{\sigma(F)} = 1$, which in turn means that $\mathrm{Card}(\sigma(F)) = \mathrm{Card}(F)$ is even.

Therefore, in W, we have

$$n_{\sigma(F)}p(\nu)T = p_\sigma T = xT$$

where $n_{\sigma(F)}T \in Y$ and $p(\nu)T \in w(\mathfrak{S}_l)$. Hence $W \subset Y \cdot w(\mathfrak{S}_l)$ and we can conclude that

$$W = Y \rtimes w(\mathfrak{S}_l) \cong (\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes \mathfrak{S}_l$$

5.4.7 Dynkin Diagram

Again we compute the Cartan integers for the root system, and so we need to compute the simple reflections $\sigma_i = \sigma_{\alpha_i}$ explicitly. We will assume that $l \geq 3$, since if $l = 1$ then the group is isomorphic to $GL(1, K)$, while if $l = 2$ then we have seen that the root system is reducible.

Suppose firstly that $1 \leq i \leq l-1$, so

$$\alpha_i : \mathrm{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i/t_{i+1}$$

We begin with the case $i < l-2$, if there is such a situation, that is, if $l \neq 3$. We apply the permutation $(i, i+1)$ to the diagonal elements t, using the formulae given in 5.4.3 to calculate the following:

$$\sigma_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } j = i \\ \alpha_{i-1} + \alpha_i & \text{if } j = i-1 \\ \alpha_i + \alpha_{i+1} & \text{if } j = i+1 \\ \alpha_j & \text{otherwise} \end{cases}$$

hence,

$$\langle \alpha_j, \alpha_i \rangle \alpha_i = \alpha_j - \sigma_i(\alpha_j) = \begin{cases} 2\alpha_i & \text{if } j = i \\ -\alpha_i & \text{if } j = i-1 \\ -\alpha_i & \text{if } j = i+1 \end{cases}$$

This σ_i inverts α_i and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_i\}$. Therefore, by Lemma 3.5.2.4, this σ_i is indeed the reflection corresponding to α_i.

We now move on to σ_{l-2}:

$$\sigma_{l-2}(\alpha_j) = \begin{cases} -\alpha_{l-2} & \text{if } j = l-2 \\ \alpha_{l-3} + \alpha_{l-2} & \text{if } j = l-3 \\ \alpha_{l-2} + \alpha_{l-1} & \text{if } j = l-1 \\ \alpha_{l-2} + \alpha_l & \text{if } j = l \\ \alpha_j & \text{otherwise} \end{cases}$$

where the first case is of course only relevant if $l > 3$. Again, this reflection inverts α_{l-1} and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_{l-2}\}$. Therefore, by Lemma 3.5.2.4, this σ_{l-2} is indeed the reflection corresponding to α_{l-2}. We therefore have

$$\langle \alpha_j, \alpha_{l-2} \rangle \alpha_{l-2} = \alpha_j - \sigma_{l-2}(\alpha_j) = \begin{cases} 2\alpha_{l-2} & \text{if } j = l-2 \\ -\alpha_{l-2} & \text{if } j = l, l-1 \\ 0 & \text{otherwise} \end{cases}$$

We now consider the reflection σ_l, that is, corresponding to the root

$$\alpha_l : \operatorname{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots t_1^{-1}) \longmapsto t_{l-1}t_l$$

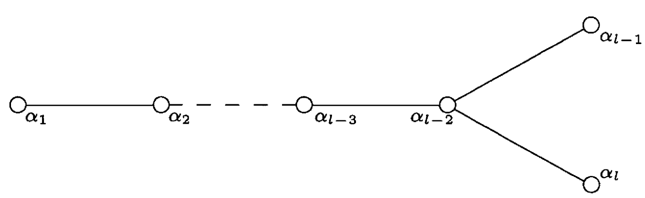
Recall that we wish to find a reflection which inverts the root α_l, but which stabilises $\Phi^+ - \{\alpha_l\}$. We show here that the correct choice is the permutation $(l-1, l+1)(l, l+2) \in \mathfrak{S}_{2l+1}$. That is, the reflection acts on elements of the torus by swapping both the $(l-1)^{\text{th}}$ and $(l+1)^{\text{th}}$, as well as the l^{th} and $(l+2)^{\text{th}}$ diagonal entries. Using the formulae in §5.4.3 again, we calculate

$$\sigma_l(\alpha_j) = \begin{cases} -\alpha_l & \text{if } j = l \\ \alpha_{l-2} + \alpha_l & \text{if } j = l-2 \\ \alpha_{l-1} + \alpha_l & \text{if } j = l-1 \\ \alpha_j & \text{otherwise} \end{cases}$$

As required, this σ_l inverts α_l and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_l\}$. Therefore, by Lemma 3.5.2.4, this σ_l is indeed the reflection corresponding to α_l. We thus get

$$\langle \alpha_j, \alpha_l \rangle \alpha_l = \alpha_j - \sigma_l(\alpha_j) = \begin{cases} 2\alpha_l & \text{if } j = l \\ -\alpha_l & \text{if } j = l-2, l-1 \\ 0 & \text{otherwise} \end{cases}$$

We now have enough data to construct the Dynkin diagram for $G = SO(2l, K)$. Evidently, there are no multiple bonds between nodes, and so there is no need to worry about arrows in the diagram. The fork at its end, however, illustrates the difficulty of the $l = 2$ case. When $l \geq 3$, however, the diagram is: Note that in the $l = 3$ case, the diagram is equivalent to that of $SL(4, K)$ up to a relabelling of its nodes.



5.5 The Odd Special Orthogonal Group

It is again convenient to assume char$K \neq 2$. The group $SO(2l+1, K)$ is defined to consist of all $x \in SL(2l+1, K)$ which satisfy

$${}^t x s x = s$$

where

$$s = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix}$$

and J is as above. Again, these polynomial conditions define a closed subgroup of $GL(2l+1, K)$, and moreover we saw in Proposition 1.3.2.8 that G is connected.

We need to use the Lie algebra $\mathscr{L}(G) = \mathfrak{g}$. It is demonstrated in §8.13.2 of [3], for example, that $\mathfrak{g}$ consists of matrices of the form

$$\mathsf{x} = \begin{pmatrix} a & w & b \\ u & 0 & -{}^t wJ \\ c & -J^t u & -J^t aJ \end{pmatrix} \tag{5.47}$$

where $a, b, c \in \text{Mat}(l, K)$ such that ${}^t b = -JbJ$ and ${}^t c = -JcJ$, that is, b and c are skew-symmetric about the skew diagonal, while u is an l-dimensional row vector and w is an l-dimensional column vector. From (5.47) we also see that $\dim G = \dim(\mathfrak{g}) = l^2 + l(l-1) + 2l = 2l^2 + l$.

5.5.1 Maximal Torus

Let $T \subset GL(2l+1, K)$ consist of matrices of the form

$$g = \begin{pmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix} \tag{5.48}$$

where $y, z \in D(l, K)$. We plan to show that T is a maximal torus in $SO(2l+1, K)$.

Suppose $g \in D(2l+1, K)$. Then g lies in $SO(2l+1, K)$ if and only if it is of determinant 1 and $gsg = s$, or, equivalently,

$$\begin{pmatrix} y & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & yJz \\ 0 & x^2 & 0 \\ zJy & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix}$$

Therefore, $g \in SO(2l+1, K)$ if and only if

1. $\det g = 1$
2. $x^2 = 1$, that is, $x = \pm 1$, and
3. $yJz = J$, that is, $y_{ii} = 1/z_{jj}$ whenever $i + j = l + 1$.

Note that the first and the third condition combine to force $x = 1$. It follows that

$$D(2l+1, K) \cap SO(2l+1, K) = \left\{ \begin{pmatrix} t_1 & & & & & & \\ & \ddots & & & & & \\ & & t_l & & & & \\ & & & 1 & & & \\ & & & & t_l^{-1} & & \\ & & & & & \ddots & \\ & & & & & & t_1^{-1} \end{pmatrix} \right\}$$

Once more we wish to calculate $C_{G'}(T)$ where $G' = GL(2l+1, K)$. The argument given in §5.4.1 can again be applied to this situation without any problems to show that $C_{G'}(T) = D(2l+1, K)$. From this we observe that $C_G(T) = G \cap C_{G'}(T) = G \cap D(2l+1, K) = T$. Suppose now that T' is a maximal torus, containing T. It is immediate that T' is also a maximal torus of $C_G(T)$, and so we get the inclusions

$$T \subset T' \subset C_G(T) = T$$

from which we conclude that $T = T'$, that is, T is a maximal torus.

5.5.2 Root System

We continue to write $G = SO(2l+1, K)$ and we take as maximal torus the group $T = G \cap D(2l+1, K)$ defined above. It is immediate that

$$\mathscr{L}(T) \subset \mathfrak{g} \cap \mathfrak{d}(2l+1, K) = \left\{ \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_l & & & & \\ & & & 0 & & & \\ & & & & -a_l & & \\ & & & & & \ddots & \\ & & & & & & -a_1 \end{pmatrix} \right\}$$

where $a_i \in K$ for $1 \le i \le l$. This last equality is simply by directly computing the intersection based on (5.47). But $\dim(\mathscr{L}(T)) = \dim T = l$, and this is clearly equal to the dimension of $\mathfrak{g} \cap \mathfrak{d}(2l+1, K)$, and so the above inclusion is equality. Moreover, Proposition 2.3.1.9 says that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathscr{L}(C_G(T))$, but this latter algebra is equal to $\mathscr{L}(T)$ since $T = C_G(T)$. Altogether, then, we have

$$\mathfrak{c}_{\mathfrak{g}}(T) = \mathscr{L}(T) = \mathfrak{g} \cap \mathfrak{d}(2l+1, K) \tag{5.49}$$

That is, the centraliser of the T-action on $\mathfrak{g}$ is equal to $\mathfrak{g} \cap \mathfrak{d}(2l+1, K)$.

We now move on to calculating the roots of G, and start by relabelling the standard basis of K^{2l+1} as follows: for $1 \le i \le l$, set $f_i = e_i$ and $f_{-i} = e_{2l+2-i}$, and finally set $f_0 = e_{l+1}$. Now let $F_{i,j}$ be the matrix which sends f_j to f_i, where $i, j \in \{0, \pm 1, \ldots, \pm l\}$. That is, $F_{i,j}$ has zeros entries everywhere except the $(i,j)^{\text{th}}$ entry, which is 1.

Now suppose $1 \le i, j \le l$, with $i \ne j$, and define the matrix $X_{\epsilon_i - \epsilon_j} = F_{i,j} - F_{-j,-i}$. That is,

$$X_{\epsilon_i - \epsilon_j} = \left(\begin{array}{c|c|c} E_{i,j} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -E_{l+1-j,l+1-i} \end{array} \right)$$

where here the $E_{r,s}$ are the elementary $l \times l$ matrices. By comparison with (5.47), we see that $X_{\epsilon_i - \epsilon_j} \in \mathfrak{g}$. Now, for any matrix $t = \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \in T$,

$$tX_{\epsilon_i - \epsilon_j} t^{-1} = \frac{t_i}{t_j} X_{\epsilon_i - \epsilon_j}$$

so $X_{\epsilon_i - \epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i / t_j$$

Now suppose $1 \le i, j \le l$, with $i \ne j$, and define the matrix $X_{\epsilon_i + \epsilon_j} = F_{i,-j} - F_{j,-i}$. That is,

$$X_{\epsilon_i + \epsilon_j} = \left(\begin{array}{c|c|c} 0 & 0 & E_{i,l+1-j} - E_{j,l+1-i} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

Again, by comparison with (5.47), we see that $X_{\epsilon_i+\epsilon_j} \in \mathfrak{g}$. For any matrix $t = \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \in T$,

$$tX_{\epsilon_i+\epsilon_j}t^{-1} = t_i t_j X_{\epsilon_i+\epsilon_j}$$

so $X_{\epsilon_i+\epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i t_j$$

Similarly, for $1 \leq i, j \leq l$, with $i \neq j$ define $X_{-\epsilon_i-\epsilon_j} = F_{-j,i} - F_{-i,j}$ for $1 \leq i, j \leq l$. That is,

$$X_{-\epsilon_i-\epsilon_j} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline E_{l+1-j,i} - E_{l+1-i,j} & 0 & 0 \end{array}\right)$$

Again, by comparison with (5.47), we see that $X_{-\epsilon_i-\epsilon_j} \in \mathfrak{g}$. For any matrix $t = \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \in T$,

$$tX_{-\epsilon_i-\epsilon_j}t^{-1} = t_i^{-1} t_j^{-1} X_{-\epsilon_i-\epsilon_j}$$

so $X_{-\epsilon_i-\epsilon_j} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \longmapsto \frac{1}{t_i t_j}$$

Now suppose $1 \leq i \leq l$. Define a matrix $X_{\epsilon_i} = F_{i,0} - F_{0,-i}$. That is,

$$X_{\epsilon_i} = \left(\begin{array}{c|c|c} 0 & e_i & 0 \\ \hline 0 & 0 & -{}^t e_{l+1-i} \\ \hline 0 & 0 & 0 \end{array}\right)$$

where here the e_r's are construed as $l \times 1$ column vectors. Again, by comparison with (5.47), we see that $X_{\epsilon_i} \in \mathfrak{g}$. For any matrix $t = \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \in T$,

$$tX_{\epsilon_i}t^{-1} = t_i X_{\epsilon_i}$$

so $X_{\epsilon_i} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i$$

Similarly, for $1 \leq i \leq l$, define a matrix $X_{-\epsilon_i} = F_{0,i} - F_{-i,0}$. That is,

$$X_{-\epsilon_i} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline {}^t e_i & 0 & 0 \\ \hline 0 & -e_{l+1-i} & 0 \end{array}\right)$$

Again, by comparison with (5.47), we see that $X_{-\epsilon_i} \in \mathfrak{g}$. For any matrix $t = \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \in T$,

$$tX_{-\epsilon_i}t^{-1} = t_i^{-1} X_{-\epsilon_i}$$

so $X_{-\epsilon_i} \in \mathfrak{g}_\alpha$, where $\alpha \in X(T)$ is the character

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1}$$

We have therefore shown that the following $2l^2$ distinct characters are roots of $G = Sp(2l, K)$:

$$\{X_{\pm(\epsilon_i - \epsilon_j)}, X_{\pm(\epsilon_i + \epsilon_j)} \mid 1 \leq i < j \leq l\} \bigcup \{X_{\pm\epsilon_i} \mid 1 \leq i \leq l\}$$

form a basis for $\mathfrak{g} - \mathfrak{t}$. It therefore follows that the roots of $SO(2l+1, K)$ are the characters:

$$\operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j^{-1}, \quad 1 \leq i, j \leq l,\ i \neq j \tag{5.50}$$

$$\operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j, \quad 1 \leq i, j \leq l,\ i \neq j \tag{5.51}$$

$$\operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}, \quad 1 \leq i, j \leq l,\ i \neq j \tag{5.52}$$

$$\operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i, \quad 1 \leq i \leq l \tag{5.53}$$

$$\operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1}, \quad 1 \leq i \leq l \tag{5.54}$$

We appeal to Lemma 3.2.3.2 once more, which says $\operatorname{Card}(\Phi) \leq \dim(G) - \dim(C_G(T)) = 2l^2 + l - l = 2l^2$, and so the above list of roots exhausts the elements of Φ. In turn we can conclude that $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$.

By comparing the spaces $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ computed above, and (5.47), we see that we have produced a decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{5.55}$$

5.5.3 Simple Roots

We begin with the case $l = 1$, that is, where $G = SO(3, K)$. We construct a homomorphism of algebraic groups $\rho : SL(2, K) \to GL(3, K)$ as follows:

$$\rho : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & -\sqrt{2}ab & -b^2 \\ \sqrt{2}cd & ad + bc & \sqrt{2}bd \\ -c^2 & \sqrt{2}cd & d^2 \end{pmatrix}$$

The fact that ρ is a homomorphism is easily checked. Moroever, by explicit calculation, one can show that $\det \rho(x) = (ad - bc)^3 = 1$ where $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, K)$, and so $\rho(SL(2, K)) \subset SL(3, K)$. Finally, it is another tedious calculation which shows that

$$\begin{pmatrix} a^2 & -\sqrt{2}ac & -c^2 \\ -\sqrt{2}ab & ad + bc & \sqrt{2}cd \\ -b^2 & \sqrt{2}bd & d^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^2 & -\sqrt{2}ab & -b^2 \\ \sqrt{2}cd & ad + bc & \sqrt{2}bd \\ -c^2 & \sqrt{2}cd & d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which shows us that $\rho(SL(2, K)) \subset SO(3, K)$. We observe that

$$\ker \rho = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, K) \,\middle|\, ad = a^2 = d^2 = 1, b = c = 0 \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Since the kernel is finite, we have $\dim \operatorname{im} \rho = \dim SL(2, K) = 3$, which is equal to the dimension of $\dim SO(3, K)$, so we conclude that ρ is surjective. It then follows from Proposition 3.2.2.9 that the root system of $SO(3, K)$ coincides with that of $SL(2, K)$.

We now move on to the case $l \geq 2$. Let $1 \leq i \leq l-1$, and define a root α_i as follows

$$\alpha_i : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i/t_{i+1}$$

and finally define a root α_l as follows:

$$\alpha_l : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_l$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. We want to show that Δ is a base for Φ, so we need to show that we can express every root $\alpha \in \Phi$ in the form $\sum c_i \alpha_i$, where the c_i's are scalars with the same sign. If we let $\alpha \in \Phi$ be a root of the form given above in (5.50), say

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i/t_j$$

where $i < j$, then

$$\alpha = \alpha_i + \cdots + \alpha_{j-1}$$

On the other hand, if $i > j$, then

$$\alpha = -\alpha_j - \cdots - \alpha_{i-1}$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.51), say

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j$$

Then

$$\alpha = (\alpha_i + \cdots + \alpha_{l-1}) + (\alpha_j + \cdots + \alpha_{l-1}) + 2\alpha_l$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.52), say

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1} t_j^{-1}$$

Then

$$\alpha = -(\alpha_i + \cdots + \alpha_{l-1}) - (\alpha_j + \cdots + \alpha_{l-1}) - 2\alpha_l$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.53), say

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i$$

Then

$$\alpha = \alpha_i + \cdots + \alpha_{l-1} + \alpha_l$$

Suppose now that $\alpha \in \Phi$ is a root of the form given above in (5.54), say

$$\alpha : \operatorname{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i^{-1}$$

Then

$$\alpha = -(\alpha_i + \cdots + \alpha_{l-1} + \alpha_l)$$

It follows that Δ is a base for Φ, that is, it is a set of simple roots. The associated set of positive roots consists of the l^2 distinct roots

$$\mathrm{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j^{-1}, \quad 1 \le i < j \le l, \tag{5.56}$$

$$\mathrm{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i t_j, \quad 1 \le i, j \le l,\ i \ne j \tag{5.57}$$

$$\mathrm{diag}(t_1, \ldots, t_l, 1, t_l^{-1}, \ldots, t_1^{-1}) \longmapsto t_i, \quad 1 \le i \le l \tag{5.58}$$

5.5.4 Radical and Unipotent Radical

In this section, we will repeat the argument given in §5.3.4 once more, so we must start by explicitly calculating the relevant Lie brackets in $\mathfrak{g}$. In particular, let $\alpha \in \Phi$, and take a nonzero element $\mathsf{x} \in \mathfrak{g}_\alpha$. Our computation of Φ in §5.5.2 above shows us that $\Phi = -\Phi$, and so $-\alpha$ is also a root of G. Take a nonzero element $\mathsf{y} \in \mathfrak{g}_{-\alpha}$. We wish to show that the bracket $[\mathsf{x}, \mathsf{y}]$ is nonzero for each $\alpha \in \Phi$.

Suppose firstly that α is of the form given in (5.50). Then, since $\dim \mathfrak{g}_\alpha = 1$, in the notation of §5.5.2, x is a nonzero multiple of $X_{\epsilon_i - \epsilon_j} = F_{i,j} - F_{-j,-i}$, with $i \ne j$, and similarly y is a nonzero multiple of $X_{\epsilon_j - \epsilon_i}$. By direct calculation, we have

$$[X_{\epsilon_i - \epsilon_j}, X_{\epsilon_j - \epsilon_i}] = F_{i,i} - F_{j,j} + F_{-j,-j} - F_{-i,-i}$$

which is not zero, since $i \ne j$. Since $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of $[X_{\epsilon_i - \epsilon_j}, X_{\epsilon_j - \epsilon_i}]$, it is also nonzero.

Suppose now that α is of the form given in (5.51). Then the fact that $\dim \mathfrak{g}_\alpha = 1$ implies that x is a nonzero multiple of $X_{\epsilon_i + \epsilon_j} = F_{i,-j} - F_{j,-i}$, with $i \ne j$, and likewise y is a nonzero multiple of $X_{-\epsilon_i, -\epsilon_j} = F_{-j,i} - F_{-i,j}$. By direct calculation,

$$[X_{\epsilon_i + \epsilon_j}, X_{-\epsilon_j - \epsilon_i}] = F_{i,i} + F_{j,j} - F_{-j,-j} - F_{-i,-i}$$

which is nonzero for all i, j. Since $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of the above expression, it, too, is nonzero. We also note that the case where α is of the form given in (5.39) is a consequence of the above calculation, since a root of this form is inverse to one of the form (5.38).

Suppose now that α is a root of the form given in (5.53). Then again the fact that $\dim \mathfrak{g}_\alpha = 1$ implies that x is a nonzero multiple of $X_{\epsilon_i} = F_{i,0} - F_{0,-i}$, and likewise y is a nonzero multiple of $X_{-\epsilon_i} = F_{0,i} - F_{-i,0}$. By direct calculation,

$$[X_{\epsilon_i}, X_{-\epsilon_i}] = F_{i,i} - F_{-i,-i}$$

which is nonzero for all i. Since $[\mathsf{x}, \mathsf{y}]$ is a nonzero multiple of the above expression, it, too, is nonzero. Finally we observe that the case where α is of the form given in (5.54), is a consequence of the above calculation, since a root of this form is inverse to one of the form (5.53). We have therefore shown that if $\mathsf{x} \in \mathfrak{g}_\alpha$ is not zero, and $\mathsf{y} \in \mathfrak{g}_{-\alpha}$ is not zero, then $[\mathsf{x}, \mathsf{y}] \ne 0$.

We now compute $R_u(G) = U$, and again we use the same strategy as in §5.3.4, namely to consider the ideal $\mathscr{L}(U) = \mathfrak{u}$. Take an element $\mathsf{x} \in \mathfrak{u}$. Since $\mathsf{x} \in \mathfrak{g}$, we can use (5.55) to express this element as $\mathsf{x} = \mathsf{x}_0 + \sum \mathsf{x}_\alpha$ where $\mathsf{x}_0 \in \mathfrak{t}$ and $\mathsf{x}_\alpha \in \mathfrak{g}_\alpha$. Moreover, we have

$$t^i.\mathsf{x} = \mathsf{x}_0 + \sum \alpha^i(t)\mathsf{x}_\alpha \in \mathfrak{u}$$

for all i and all $t \in T$. Again we note that given any distinct roots $\alpha, \beta \in \Phi$, we can choose an element $t \in T$ such that $\alpha^i(t) \neq \beta^i(t)$ for all $i > 0$. We can then repeat the argument given in §5.3.4, to show that $\mathsf{x}_0 \in \mathfrak{u}$, and $\mathsf{x}_\alpha \in \mathfrak{u}$ for all $\alpha \in \Phi$.

Therefore $\mathsf{x}_0 \in \mathfrak{u} \cap \mathfrak{t}$. But $\mathfrak{u}$ consists of nilpotent elements, and $\mathfrak{t}$ consists of semisimple elements, and so $\mathsf{x}_0 = 0$. On the other hand, we have $\mathsf{x}_\alpha \in \mathfrak{u} \cap \mathfrak{g}_\alpha$, so suppose this element is nonzero. Select a nonzero element $\mathsf{y} \in \mathfrak{g}_{-\alpha}$, and so Lemma 3.4.3.1 tells us that $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{g}_0 = \mathfrak{t}$. On the other hand, we observed above that $\mathfrak{u}$ is an ideal, and so $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{u}$ since $\mathsf{x}_\alpha \in \mathfrak{u}$. We have shown, then, that $[\mathsf{x}_\alpha, \mathsf{y}] \in \mathfrak{u} \cap \mathfrak{t}$, and so, being both semisimple and nilpotent, we are forced to conclude that $[\mathsf{x}_\alpha, \mathsf{y}] = 0$. But this contradicts the argument which began this section, and so it follows that $\mathsf{x}_\alpha = 0$ for all $\alpha \in \Phi$.

We have therefore shown that $\mathfrak{u} = \{0\}$. In particular, $0 = \dim \mathfrak{u} = \dim U$. But U is connected, and so must be trivial, and thus G is reductive.

Finally, we can compute the radical in the now familiar way. Since G is reductive, we can use Theorem 3.4.3.2(6), which says $Z(G)^\circ = \left(\bigcap_{\alpha\in\Phi} T_\alpha\right)^\circ$. Suppose, then, that $t = \operatorname{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1}) \in T$ is such that $\alpha(t) = 1$ for all $\alpha \in \Phi$. In particular, this condition on roots of the form (5.53) forces $t_i = 1$ for all i. It follows that $Z(G)$ is trivial, but since G is reductive, we can apply Lemma 3.1.1.9 which tells us $R(G) = Z(G)^\circ$, and so we conclude $R(G)$ is trivial, and therefore G is semisimple.

5.5.5 Borel Subgroup

Let α be a root of the form (5.56). Define a group $U_\alpha \subset G$ which consists of elements of the form

$$\begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & J^t u^{-1} J \end{pmatrix}$$

where $u \in U_{ij}$, the unipotent group consisting of elements with arbitrary $(i,j)^{\text{th}}$ entry, 1's on the diagonal, and zeros everywhere else. Now U_α is a subgroup of G, and moreover, since $i < j$, U_{ij} is upper triangular, and thus so is U_α. We turn to the Lie algebra of U_α. A straightforward calculation shows that, using the notation of §5.5.2, $X_{\epsilon_i - \epsilon_j} \in \mathscr{L}(U_\alpha)$, and since $\dim U_\alpha = 1$, it follows that $\mathscr{L}(U_\alpha) = \mathfrak{g}_\alpha$. By Theorem 3.4.5.1, U_α is the unique connected unipotent subgroup of G having α as a root.

Suppose now that α is a root of the form (5.57). Define a group $U_\alpha \subset G$

consisting of elements of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where a is an $l \times l$ matrix with $a_{i,l+1-j} = -a_{j,l+1-i}$ and all other entries equal to 0. Evidently a is skew symmetric about its skew diagonal, that is, $a = -J^t a J$, and this is enough to guarantee that U_α is a subset of G, a fact which can be shown by evaluating ${}^t x s x$ for $x \in U_\alpha$. Moreover,

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a+b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so, since $(a+b)_{i,l+1-j} = -(a+b)_{j,l+1-i}$, it follows that U_α is closed under multiplication and taking of inverses, and so is a subgroup of G. Moreover, a relatively simple calculation shows that $X_{\epsilon_i+\epsilon_j} \in \mathscr{L}(U_\alpha)$, and so again we see that U_α is the unique connected unipotent subgroup of G which has Lie algebra equal to $\mathfrak{g}_\alpha$.

Finally, we address the case where α is a root of the form (5.58). Define a subset $U_\alpha \subset G$ consisting of elements of the form

$$\begin{pmatrix} I_l & v & -\frac{1}{2}v^t v \\ 0 & 1 & -{}^t v \\ 0 & 0 & I_l \end{pmatrix} \tag{5.59}$$

where I_l denotes the $l \times l$ identity matrix, and v is an l-dimensional column vector with arbitrary i^{th} coordinate v_i, and all other coordinates equal to 0. The upper right submatrix is therefore the $l \times l$ matrix with $(i,i)^{\text{th}}$ coordinate equal to $-\frac{1}{2}v_i^2$. Again, we can evaluate ${}^t x s x$ for elements $x \in U_\alpha$ to show that U_α is indeed a subset of G. Moreover,

$$\begin{aligned}
\begin{pmatrix} I_l & v & -\frac{1}{2}v^t v \\ 0 & 1 & -{}^t v \\ 0 & 0 & I_l \end{pmatrix} \begin{pmatrix} I_l & w & -\frac{1}{2}v^t v \\ 0 & 1 & -{}^t w \\ 0 & 0 & I_l \end{pmatrix} &= \begin{pmatrix} I_l & v+w & -\frac{1}{2}v^t v - \frac{1}{2}w^t w + v^t w \\ 0 & 1 & -{}^t v - {}^t w \\ 0 & 0 & I_l \end{pmatrix} \\
&= \begin{pmatrix} I_l & (v+w) & -\frac{1}{2}(v+w)^t(v+w) \\ 0 & 1 & -{}^t(v+w) \\ 0 & 0 & I_l \end{pmatrix}
\end{aligned}$$

and so U_α is a group. Again we can verify that $X_{\epsilon_i} \in \mathscr{L}(U_\alpha)$, and so U_α is the unique connected unipotent subgroup with α as a root.

It follows that the group B which is generated by the groups U_α as described above contains all the positive roots, and is therefore a Borel subgroup, namely the Borel subgroup $B(\Delta)$. Moreover, since B is upper triangular, it is a subgroup of $G \cap T(2l+1, K)$. But this latter group is itself solvable, being a subgroup of $T(2l+1, K)$, and so, since B is maximal, it follows that $B = G \cap T(2l+1, K)$.

5.5.6 Weyl Group

We first have to calculate the normaliser $N_G(T)$ for $T = G \cap D(2l+1, K)$. Again, by an similar argument to that of §5.3.6, it turns out that $N_G(T) = G \cap N_{GL(2l,K)}\big(D(2l+1,K)\big)$.

We now turn to the computation of $W = N_G(T)/T$. For each $\sigma \in \mathfrak{S}_{l+1}$, define the matrix $p(\sigma)$ in $GL(2l+1, K)$ as follows:

$$p(\sigma) = \begin{pmatrix} p_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_\pi p_\sigma p_\pi \end{pmatrix}$$

where $J = p_\pi$ for the permutation $\pi = (1,l)(2,l-1)\cdots$. Again, as in §5.3.6, it is easy to show that $p(\sigma) \in N_G(T)$ for all $\sigma \in \mathfrak{S}_l$, and so we can define an injective map

$$\begin{aligned} w : \mathfrak{S}_l &\longrightarrow W \\ \sigma &\longmapsto p(\sigma)T \end{aligned}$$

We turn now to the remaining elements of W.

We use the same basis for K^{2l+1} as was used in §5.5.2 above: for $1 \le i \le l$, set $f_i = e_i$ and $f_{-i} = e_{2l+2-i}$, and finally set $f_0 = e_{l+1}$. We now define our monomial matrices $n_i \in GL(2l+1, K)$, for $1 \le i \le l$:

$$\begin{aligned} n_i f_k &= f_k, && \text{if } 1 \le k \le l,\ k \ne i \\ n_i f_{-k} &= f_{-k}, && \text{if } 1 \le k \le l,\ k \ne i \\ n_i f_0 &= -f_0, \\ n_i f_i &= f_{-i}, \\ n_i f_{-i} &= f_i \end{aligned} \tag{5.60}$$

Then $\det(n_i) = 1$ and again we have $n_i^{-1} = {}^t n_i = n_i$, and $n_i n_j = n_j n_i$ for all $i, j \in \{1, \ldots, l\}$.

Using similar arguments to that in §5.3.6, it can be shown that ${}^t n_i s n_i = s$, and that $n_i t n_i$ is diagonal for all $t \in T$, so $n_i \in N_G(T)$. We go on to define a matrix

$$n_F = \prod_{i \in F} n_i$$

for $F \subset \{1, \ldots, l\}$. Again, as in §5.3.6, we define a subset Y of W as follows:

$$Y = \big\{ n_F T \,|\, F \subset \{1, \ldots, l\} \big\}$$

Once more Y is a normal subgroup of W, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^l$ under the same group isomorphism q as in §5.3.6.

It remains to show that $W = Y \rtimes w(\mathfrak{S}_l)$. But the argument is really the same as in §5.3.6. We first show that $Y \cap w(\mathfrak{S}_l)$ is trivial, in the usual way. We then consider an element $x = p_\sigma T \in W$ for $\sigma \in \mathfrak{S}_{2l+1}$ such that $\sigma\gamma = \gamma\sigma$. Note, moreover, that if $x \in G$ then there is an additional condition on σ, namely that $\sigma(l+1) = l+1$. From here we can proceed as usual, expressing x in the form

$$x = n_{\sigma(F)} p(\nu) T$$

for a permutation $\nu \in \mathfrak{S}_l$, which is equal to $\mu\sigma$ on $\{1, \dots, l\}$, and where F and μ are defined as in §5.3.6. The conclusion is

$$W = Y \rtimes w(\mathfrak{S}_l) \cong (\mathbb{Z}/2\mathbb{Z})^l \rtimes \mathfrak{S}_l$$

5.5.7 Dynkin Diagram

Again we compute the Cartan integers for the root system, and so we need to compute the simple reflections $\sigma_i = \sigma_{\alpha_i}$ explicitly. We saw in §5.5.3 that the case $l = 1$ corresponds to that of $SL(2, K)$, so we assume that $l \geq 2$.

Suppose firstly that $1 \leq i \leq l-1$, so

$$\alpha_i : \mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1}) \longmapsto t_i / t_{i+1}$$

Once more we begin with the case $i < l-1$. We apply the permutation $(i, i+1)$ to the diagonal elements t, using the formulae given in 5.5.3 to calculate the following:

$$\sigma_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } j = i \\ \alpha_{i-1} + \alpha_i & \text{if } j = i-1 \\ \alpha_i + \alpha_{i+1} & \text{if } j = i+1 \\ \alpha_j & \text{otherwise} \end{cases}$$

This σ_i inverts α_i and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_i\}$. Therefore, by Lemma 3.5.2.4, this σ_i is indeed the reflection corresponding to α_i. Hence,

$$\langle \alpha_j, \alpha_i \rangle \alpha_i = \alpha_j - \sigma_i(\alpha_j) = \begin{cases} 2\alpha_i & \text{if } j = i \\ -\alpha_i & \text{if } j = i-1 \\ -\alpha_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

For σ_{l-1}:

$$\sigma_{l-1}(\alpha_j) = \begin{cases} -\alpha_{l-1} & \text{if } j = l-1 \\ \alpha_{l-1} + \alpha_l & \text{if } j = l \\ \alpha_j & \text{otherwise} \end{cases}$$

Again, this reflection inverts α_{l-1} and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_{l-1}\}$. Therefore, by Lemma 3.5.2.4, this σ_{l-1} is indeed the reflection corresponding to α_{l-1}. We therefore get

$$\langle \alpha_j, \alpha_{l-1} \rangle \alpha_{l-1} = \alpha_j - \sigma_{l-1}(\alpha_j) = \begin{cases} 2\alpha_{l-1} & \text{if } j = l-1 \\ -\alpha_{l-1} & \text{if } j = l \\ 0 & \text{otherwise} \end{cases}$$

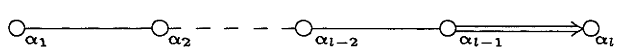
We now consider the reflection σ_l. The situation is similar to that of §5.3.7. Since $\alpha_l(t) = t_l$, the reflection which inverts this root but which stabilises $\Phi^+ - \{\alpha_l\}$ is evidently the transposition $(l, l+2) \in \mathfrak{S}_{2l+1}$. That is, the reflection acts on elements of the torus $t = \mathrm{diag}\{t_1, \ldots, t_l, 1, t_l^{-1}, \ldots\}$ by swapping the l^{th} and the $(l+2)^{\text{th}}$ diagonal entries. Using the formulae in §5.5.3 again, we calculate

$$\sigma_l(\alpha_j) = \begin{cases} -\alpha_l & \text{if } j = l \\ \alpha_{l-1} + 2\alpha_l & \text{if } j = l-1 \\ \alpha_j & \text{otherwise} \end{cases}$$

Again, this reflection does indeed invert α_i and also sends the remaining simple roots to the set $\Phi^+ - \{\alpha_l\}$. Therefore, by Lemma 3.5.2.4, this σ_l is indeed the reflection corresponding to α_l. and so

$$\langle \alpha_j, \alpha_l \rangle \alpha_l = \alpha_j - \sigma_l(\alpha_j) = \begin{cases} 2\alpha_l & \text{if } j = l \\ -2\alpha_l & \text{if } j = l-1 \\ 0 & \text{otherwise} \end{cases}$$

We now have enough data to construct the Dynkin diagram for $G = SO(2l+1, K)$. It is very similar to that of $Sp(2l, K)$, because the Cartan integers almost correspond exactly. The one difference comes from the fact that in this case, $\langle \alpha_{l-1}, \alpha_l \rangle \big(\langle \alpha_l, \alpha_{l-1} \rangle\big)^{-1} = 2$, and so there is an arrow pointing from the node α_{l-1} to the node α_l. Evidently, this makes no difference in the case $l = 2$. The Dynkin diagram is:



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